

QUATERNIONIC KÄHLER AND $\text{Spin}(7)$ METRICS ARISING FROM QUATERNIONIC CONTACT EINSTEIN STRUCTURES

L.C. DE ANDRÉS, M. FERNÁNDEZ, S. IVANOV, J.A. SANTISTEBAN, L. UGARTE, AND D. VASSILEV

ABSTRACT. We construct left invariant quaternionic contact (qc) structures on Lie groups with zero and non-zero torsion and with non-vanishing quaternionic contact conformal curvature tensor, thus showing the existence of non-flat quaternionic contact manifolds. We prove that the product of the real line with a seven dimensional manifold, equipped with a certain qc structure, has a quaternionic Kähler metric as well as a metric with holonomy contained in $\text{Spin}(7)$. As a consequence we determine explicit quaternionic Kähler metrics and $\text{Spin}(7)$ -holonomy metrics which seem to be new. Moreover, we give explicit non-compact eight dimensional almost quaternion hermitian manifolds with either a closed fundamental four form or fundamental two forms defining a differential ideal that are not quaternionic Kähler.

CONTENTS

| | |
|--|----|
| 1. Introduction | 2 |
| 2. Quaternionic contact manifolds | 3 |
| 2.1. Quaternionic contact structures and the Biquard connection | 3 |
| 2.2. Torsion and curvature | 5 |
| 2.3. The qc conformal curvature | 6 |
| 3. Explicit examples of quaternionic contact structures | 6 |
| 3.1. Example 0: The quaternionic Heisenberg group $G(\mathbb{H})$ - the Biquard-flat qc structure | 6 |
| 3.2. Example 1: zero torsion qc-flat structure. | 7 |
| 3.3. Example 2: zero torsion qc-non-flat structure. | 9 |
| 3.4. Example 3: non-zero torsion qc-non-flat structure. | 10 |
| 4. $Sp(n)Sp(1)$ -hypo structures and quaternionic Kähler manifolds | 10 |
| 4.1. Quaternionic contact and $Sp(n)Sp(1)$ -hypo structures | 12 |
| 4.2. Construction of quaternionic Kähler structures using qc structures | 13 |
| 4.3. Quaternionic Kähler metrics based on qc Einstein structure with zero qc scalar curvature | 14 |
| 4.4. Quaternionic Kähler metrics based on a qc Einstein structure with negative qc scalar curvature | 15 |
| 4.5. Quaternionic Kähler metrics arising from a 3-Sasakian structure | 17 |
| 4.6. Non quaternionic Kähler structures with closed four form in dimension 8 | 17 |
| 4.7. Eight dimensional non quaternionic Kähler structures with fundamental forms generating a differential ideal | 18 |
| 5. $Sp(1)Sp(1)$ structures and $\text{Spin}(7)$ -holonomy metrics | 19 |
| 5.1. Construction of $\text{Spin}(7)$ -holonomy metrics using qc structures | 19 |
| 5.2. $\text{Spin}(7)$ -holonomy metrics based on qc Einstein structure with zero qc scalar curvature | 21 |
| 5.3. $\text{Spin}(7)$ -holonomy metrics based on qc Einstein structure with negative scalar curvature | 21 |
| 5.4. $\text{Spin}(7)$ -holonomy metrics from a 3-Sasakian manifold | 22 |
| References | 22 |

Date: September 15, 2010.

2000 Mathematics Subject Classification. 58J60, 53C26.

Key words and phrases. quaternionic contact structures, Einstein structures, qc conformal flatness, qc conformal curvature, quaternionic Kähler structures, $\text{Spin}(7)$ -holonomy metrics, quaternionic Kähler and hyper Kähler metrics.

This work has been partially funded by grant MCINN (Spain) MTM2008-06540-C02-01/02.

1. INTRODUCTION

It is well known that the sphere at infinity of a non-compact symmetric space M of rank one carries a natural Carnot-Carathéodory structure (see [28, 31]). Quaternionic contact structures were introduced by Biquard in [5, 6], and they appear naturally as the conformal boundary at infinity of quaternionic Kähler spaces. Such structures are also relevant for the quaternionic contact Yamabe problem which is naturally connected with the extremals and the best constant in an associated Sobolev-type (Folland-Stein [14]) embedding on the quaternionic Heisenberg group [36, 20, 21].

Following Biquard, [5, 6], quaternionic contact structure (*qc structure*) on a real $(4n + 3)$ -dimensional manifold M is a codimension three distribution H locally given as the kernel of a \mathbb{R}^3 -valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$, such that, the three 2-forms $d\eta_i|_H$ are the (local) fundamental forms of a quaternionic structure on H . The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on \mathbb{R}^3 , and therefore H is equipped with a conformal class $[g]$ of Riemannian metrics. The transformations preserving a given qc structure η , i.e. $\bar{\eta} = \mu\Psi\eta$ for a positive smooth function μ and a non-constant $SO(3)$ matrix Ψ are called *quaternionic contact conformal (qc conformal for short) transformations*. If the function μ is constant we have *qc-homothetic transformations*. To every metric in the fixed conformal class $[g]$ on H one can associate a linear connection preserving the qc structure, see [5], which we shall call the Biquard connection. This connection is invariant under qc homothetic transformations but changes in a non-trivial way under qc conformal transformations. The torsion endomorphism of the Biquard connection is the obstruction for a qc structure to be locally qc homothetic to a 3-Sasakian [20, 24, 22] and its vanishing is equivalent to the vanishing of the trace-free part of the horizontal qc-Ricci forms, i.e. to the condition that the qc structure is qc Einstein [20].

The quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ with its standard left-invariant qc structure is locally the unique (up to a $SO(3)$ -action) example of a qc structure with flat Biquard connection [20]. In fact, the vanishing of the curvature on H implies the flatness of the Biquard connection [23]. The quaternionic Cayley transform is a qc conformal equivalence between the standard 3-Sasakian structure on the $(4n + 3)$ -dimensional sphere S^{4n+3} minus a point and the flat qc structure on $\mathbf{G}(\mathbb{H})$ [20]. All qc structures locally qc conformal to $\mathbf{G}(\mathbb{H})$ and S^{4n+3} are characterized in [23] by the vanishing of a tensor invariant, the qc-conformal curvature W^{qc} defined in terms of the curvature and torsion of the Biquard connection.

Examples of qc manifolds arising from quaternionic Kähler deformations are given in [5, 6, 12]. Duchemin shows [12] that for any qc manifold there exists a quaternionic Kähler manifold such that the qc manifold is realized as a hypersurface. However, the embedding in his construction is not isometric and it is difficult to write an explicit expression of the quaternionic Kähler metric except the 3-Sasakian case where the cone metric is hyperKähler.

One purpose of this paper is to find new explicit examples of qc structures. We construct explicit left invariant qc structures on three Lie groups of dimension seven (that we call L_1 , L_2 and L_3) with zero and non-zero torsion endomorphism of the Biquard connection for which the qc-conformal curvature tensor does not vanish, $W^{qc} \neq 0$, thus showing the existence of qc manifolds not locally qc conformal to the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$. We present a left invariant qc structure with zero torsion endomorphism of the Biquard connection on a seven dimensional non-nilpotent Lie group L_1 . Surprisingly, we obtain that this qc structure is locally qc conformal to the flat qc structure on the two-step nilpotent quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ showing that the qc conformal curvature is zero and applying the main result in [23]. Consequently, this fact yields the existence of a local function μ such that the qc conformal transformation $\bar{\eta} = \mu\eta$ preserves the vanishing of the torsion of the Biquard connection.

The second goal of the paper is to construct explicit quaternionic Kähler and $Spin(7)$ -holonomy metrics, i.e. metrics with holonomy contained in $Sp(n)Sp(1)$ and $Spin(7)$, respectively, on a product of a qc manifold with a real line. We generalize the notion of a qc structure, namely, we define an $Sp(n)Sp(1)$ -hypo structure on a $(4n + 3)$ -dimensional manifold as the structure induced on an orientable hypersurface of a quaternionic Kähler manifold. We show that a qc structure is an $Sp(n)Sp(1)$ -hypo precisely when its fundamental four form, defined by (4.8), vanishes. In Theorem 4.9, we prove that there is a quaternionic Kähler structure on the product of the real line with a smooth qc Einstein manifold of dimension bigger than seven. In dimension seven, we show that the product of a real line with a qc Einstein structure of constant qc scalar curvature

has a quaternionic Kähler metric (Theorem 4.9) as well as a metric with holonomy contained in $\text{Spin}(7)$ (Theorem 5.3). The construction and the properties of the obtained metrics with special holonomy depends on the sign (or the vanishing) of the qc scalar curvature. In the negative qc scalar curvature case, we present explicit quaternionic Kähler metrics, complete as sub-Riemannian metrics, and $\text{Spin}(7)$ -holonomy metrics on the product of the seven dimensional Lie groups L_1 and L_2 with the real line, some of which seem to be new. In the case of zero qc scalar curvature, using the quaternionic Heisenberg group, we rediscover the complete Einstein metric on an eight-dimensional solvable Lie group constructed by Gibbons et al in [17] as an Einstein metric starting with a T^3 bundle over T^4 , [17, equation (148)]. Thus, we show that the Einstein metric in dimension eight discovered in [17] is in fact a quaternionic Kähler metric and extend it to obtain complete quaternionic Kähler metrics on a $4n+4$ dimensional solvable Lie groups constructed on $G(\mathbb{H}) \times \mathbb{R}$. In the positive qc scalar curvature case we give a general construction which includes well known metrics on the product of a 3-Sasakian manifold with the real line.

It is well known that in dimension eight an almost quaternion hermitian structure with closed fundamental four form is not necessarily quaternionic Kähler [33]. This fact was confirmed by Salamon constructing in [32] a compact example of an almost quaternion hermitian manifold with closed fundamental four form which is not Einstein, and therefore it is not a quaternionic Kähler. We give a three parameter family of explicit non-compact eight dimensional almost quaternion hermitian manifolds with closed fundamental four form which are not quaternionic Kähler. We also check that these examples are not Einstein.

To the best of our knowledge there is no known example of an almost quaternion hermitian eight dimensional manifold with closed fundamental four form which is Einstein but not quaternionic Kähler.

Finally, we obtain an explicit family of almost quaternion hermitian structures such that the fundamental 2-forms define a differential ideal but the structure is not a quaternionic Kähler manifold, see also [27] for earlier examples.

Convention 1.1.

- a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$;
- b) $\{e_1, \dots, e_{4n}\}$ denotes a local orthonormal basis of the horizontal space H ;
- c) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
- d) s will be any number from the set $\{1, 2, 3\}$, $s \in \{1, 2, 3\}$.

Acknowledgments We thank Charles Boyer for useful conversations leading to Remark 3.1.

The research was initiated during the visit of the third author to the Abdus Salam ICTP, Trieste as a Senior Associate, Fall 2008. He also thanks ICTP for providing the support and an excellent research environment. S.I. is partially supported by the Contract 082/2009 with the University of Sofia ‘St.Kl.Ohridski’. S.I and D.V. are partially supported by Contract ‘Idei’, DO 02-257/18.12.2008 and DID 02-39/21.12.2009. This work has been also partially supported through grant MCINN (Spain) MTM2008-06540-C02-01/02.

2. QUATERNIONIC CONTACT MANIFOLDS

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [5], [20] and [23] which we will use in this paper.

2.1. Quaternionic contact structures and the Biquard connection. A quaternionic contact (qc) manifold (M, g, \mathbb{Q}) is a $4n+3$ -dimensional manifold M with a codimension three distribution H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 . In addition H has an $Sp(n)Sp(1)$ structure, that is, it is equipped with a Riemannian metric g and a rank-three bundle \mathbb{Q} consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1 I_2 = -I_2 I_1 = I_3$, $I_1 I_2 I_3 = -id|_H$ which are hermitian compatible with the metric $g(I_s \cdot, I_s \cdot) = g(\cdot, \cdot)$ and the following compatibility condition holds $2g(I_s X, Y) = d\eta_s(X, Y)$, $X, Y \in H$.

A special phenomena, noted in [5], is that the contact form η determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

Correspondingly, given a qc manifold we shall denote with η any associated contact form. The associated contact form is determined up to an $SO(3)$ -action, namely if $\Psi \in SO(3)$ then $\Psi\eta$ is again a contact form satisfying the above compatibility condition (rotating also the almost complex structures). On the other

hand, if we consider the conformal class $[g]$ on H , the associated contact forms are determined up to a multiplication with a positive conformal factor μ and an $SO(3)$ -action, namely if $\Psi \in SO(3)$ then $\mu\Psi\eta$ is a contact form associated with a metric in the conformal class $[g]$ on H . A qc manifold (M, \bar{g}, \mathbb{Q}) is called qc conformal to (M, g, \mathbb{Q}) if $\bar{g} \in [g]$. In that case, if $\bar{\eta}$ is a corresponding associated 1-form with complex structures \bar{I}_s , $s = 1, 2, 3$, we have $\bar{\eta} = \mu\Psi\eta$ for some $\Psi \in SO(3)$ with smooth functions as entries and a positive function μ . In particular, starting with a qc manifold (M, η) and defining $\bar{\eta} = \mu\eta$ we obtain a qc manifold $(M, \bar{\eta})$ qc conformal to the original one.

If the first Pontryagin class of M vanishes then the 2-sphere bundle of \mathbb{R}^3 -valued 1-forms is trivial [2], i.e. there is a globally defined form η that annihilates H , we denote the corresponding qc manifold (M, η) . In this case the 2-sphere of associated almost complex structures is also globally defined on H .

On a qc manifold with a fixed metric g on H there exists a canonical connection defined in [5] when the dimension $(4n + 3) > 7$, and in [11] for the 7-dimensional case. We have

Theorem 2.1. [5] *Let (M, g, \mathbb{Q}) be a qc manifold of dimension $4n + 3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM , such that:*

- i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H , i.e. $\nabla g = 0, \nabla \sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$;
- ii) for $\xi \in V$, the endomorphism $T(\xi, \cdot)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;
- iii) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H , i.e. $\nabla \varphi = 0$.

In ii), the inner product \langle, \rangle of $End(H)$ is given by $\langle A, B \rangle = \sum_{i=1}^{4n} g(A(e_i), B(e_i))$, for $A, B \in End(H)$.

We shall call the above connection *the Biquard connection*. Biquard [5] also described the supplementary subspace V , namely V is (locally) generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$(2.1) \quad \begin{aligned} \eta_s(\xi_k) &= \delta_{sk}, & (\xi_s \lrcorner d\eta_s)_{|H} &= 0, \\ (\xi_s \lrcorner d\eta_k)_{|H} &= -(\xi_k \lrcorner d\eta_s)_{|H}, \end{aligned}$$

where \lrcorner denotes the interior multiplication. The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields.

If the dimension of M is seven, there might be no vector fields satisfying (2.1). Duchemin shows in [11] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then Theorem 2.1 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).

Notice that equations (2.1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend g to a metric on M by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_s, \xi_k) = \delta_{sk}$. The extended metric does not depend on the action of $SO(3)$ on V , but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on TM , $\nabla g = 0$. Since the Biquard connection is metric it is connected with the Levi-Civita connection ∇^g of the metric g by the general formula

$$(2.2) \quad g(\nabla_A B, C) = g(\nabla_A^g B, C) + \frac{1}{2} [g(T(A, B), C) - g(T(B, C), A) + g(T(C, A), B)].$$

The covariant derivative of the qc structure with respect to the Biquard connection and the covariant derivative of the distribution V are given by $\nabla I_i = -\alpha_j \otimes I_i + \alpha_k \otimes I_j$, $\nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j$. The $sp(1)$ -connection 1-forms α_s on H are expressed in [5] by

$$(2.3) \quad \alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V,$$

while the $sp(1)$ -connection 1-forms α_s on the vertical space V are calculated in [20]

$$(2.4) \quad \alpha_i(\xi_s) = d\eta_s(\xi_j, \xi_k) - \delta_{is} \left(\frac{S}{2} + \frac{1}{2} (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2)) \right),$$

where S is the *normalized* qc scalar curvature defined below in (2.5). The vanishing of the $sp(1)$ -connection 1-forms on H implies the vanishing of the torsion endomorphism of the Biquard connection (see [20]).

The fundamental 2-forms $\omega_i, i = 1, 2, 3$ [5] are defined by $2\omega_i|_H = d\eta_i|_H$, $\xi \lrcorner \omega_i = 0$, $\xi \in V$. The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi =$

$T(\xi, \cdot) : H \rightarrow H$, $\xi \in V$. Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part T_ξ^0 and skew-symmetric part $b_\xi, T_\xi = T_\xi^0 + b_\xi$. O. Biquard shows in [5] that the torsion T_ξ is completely trace-free, $tr T_\xi = tr T_\xi \circ I_s = 0$, its symmetric part has the properties $T_{\xi_i}^0 I_i = -I_i T_{\xi_i}^0$, $I_2(T_{\xi_2}^0)^{+-+} = I_1(T_{\xi_1}^0)^{-+-}$, $I_3(T_{\xi_3}^0)^{-+-} = I_2(T_{\xi_2}^0)^{-++}$, $I_1(T_{\xi_1}^0)^{-++} = I_3(T_{\xi_3}^0)^{+-+}$, where where the superscript $+++$ means commuting with all three I_i , $+-+$ indicates commuting with I_1 and anti-commuting with the other two and etc. The skew-symmetric part can be represented as $b_{\xi_i} = I_i u$, where u is a traceless symmetric $(1,1)$ -tensor on H which commutes with I_1, I_2, I_3 . If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T_\xi^0$.

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc scalar curvature (see (2.5)) is a positive constant [20]. We remind that a $(4n+3)$ -dimensional Riemannian manifold (M, g) is called 3-Sasakian if the cone metric $g_c = t^2 g + dt^2$ on $C = M \times \mathbb{R}^+$ is a hyper Kähler metric, namely, it has holonomy contained in $Sp(n+1)$ [8]. A 3-Sasakian manifold of dimension $(4n+3)$ is Einstein with positive Riemannian scalar curvature $(4n+2)(4n+3)$ [25] and if complete it is compact with a finite fundamental group, (see [7] for a nice overview of 3-Sasakian spaces).

2.2. Torsion and curvature. Let $R = [\nabla, \nabla] - \nabla[\cdot, \cdot]$ be the curvature tensor of ∇ and the dimension is $4n+3$. We denote the curvature tensor of type $(0,4)$ by the same letter, $R(A, B, C, D) := g(R(A, B)C, D)$, $A, B, C, D \in \Gamma(TM)$. The Ricci 2-forms and the normalized scalar curvature of the Biquard connection, called *qc-Ricci forms* ρ_s and *normalized qc-scalar curvature* S , respectively, are defined by

$$(2.5) \quad 4n\rho_s(A, B) = R(A, B, e_a, I_s e_a), \quad 8n(n+2)S = R(e_b, e_a, e_a, e_b).$$

The $sp(1)$ -part of R is determined by the Ricci 2-forms and the connection 1-forms by

$$(2.6) \quad R(A, B, \xi_i, \xi_j) = 2\rho_k(A, B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A, B), \quad A, B \in \Gamma(TM).$$

The two $Sp(n)Sp(1)$ -invariant trace-free symmetric 2-tensors $T^0(X, Y) = g((T_{\xi_1}^0 I_1 + T_{\xi_2}^0 I_2 + T_{\xi_3}^0 I_3)X, Y)$, $U(X, Y) = g(uX, Y)$ on H , introduced in [20], have the properties:

$$(2.7) \quad \begin{aligned} T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) &= 0, \\ U(X, Y) &= U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y). \end{aligned}$$

In dimension seven ($n = 1$), the tensor U vanishes identically, $U = 0$.

We shall need the following identity taken from [23, Proposition 2.3]

$$(2.8) \quad 4g(T^0(\xi_s, I_s X), Y) = T^0(X, Y) - T^0(I_s X, I_s Y)$$

Definition 2.2. A qc structure is said to be qc Einstein if the horizontal qc-Ricci 2-forms are scalar multiple of the fundamental 2-forms,

$$\rho_s(X, Y) = \nu_s \omega_s(X, Y).$$

For a qc Einstein structure the functions ν_s are all equal and can be expressed as a constant multiple of the qc scalar curvature [20].

The horizontal Ricci 2-forms can be expressed in terms of the torsion of the Biquard connection [20] (see also [21, 23]). We collect the necessary facts from [20, Theorem 1.3, Theorem 3.12, Corollary 3.14, Proposition 4.3 and Proposition 4.4] with slight modification presented in [23]

Theorem 2.3. [20] On a $(4n+3)$ -dimensional qc manifold (M, η, \mathbb{Q}) the next formulas hold

$$(2.9) \quad \begin{aligned} \rho_i(X, I_l Y) &= -\frac{1}{2} [T^0(X, Y) + T^0(I_l X, I_l Y)] - 2U(X, Y) - Sg(X, Y), \\ T(\xi_i, \xi_j) &= -S\xi_k - [\xi_i, \xi_j]_H, \quad S = -g(T(\xi_1, \xi_2), \xi_3) \\ g(T(\xi_i, \xi_j), X) &= -\rho_k(I_i X, \xi_i) = -\rho_k(I_j X, \xi_j) = -g([\xi_i, \xi_j], X), \quad \rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j) = \frac{1}{2}\xi_j(S); \\ \rho_i(X, \xi_i) &= -\frac{X(S)}{4} + \frac{1}{2} (-\rho_i(\xi_j, I_k X) + \rho_j(\xi_k, I_i X) + \rho_k(\xi_i, I_j X)). \end{aligned}$$

For $n = 1$ the above formula holds with $U = 0$.

The qc Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case S is constant and the vertical distribution is integrable provided $n > 1$.

2.3. The qc conformal curvature. The qc conformal curvature tensor W^{qc} , introduced in [23], is the obstruction for a qc structure to be locally qc conformal to the flat structure on the quaternionic Heisenberg group $G(\mathbb{H})$. Denote $L_0 = \frac{1}{2}T^0 + U$, the tensor W^{qc} can be expressed by [23]

$$(2.10) \quad W^{qc}(X, Y, Z, V) = R(X, Y, Z, V) + (g \oslash L_0)(X, Y, Z, V) + \sum_{s=1}^3 (\omega_s \oslash I_s L_0)(X, Y, Z, V) \\ - \frac{1}{2} \sum_{s=1}^3 \left[\omega_s(X, Y) \left\{ T^0(Z, I_s V) - T^0(I_s Z, V) \right\} + \omega_s(Z, V) \left\{ T^0(X, I_s Y) - T^0(I_s X, Y) - 4U(X, I_s Y) \right\} \right] \\ + \frac{S}{4} \left[(g \oslash g)(X, Y, Z, V) + \sum_{s=1}^3 \left((\omega_s \oslash \omega_s)(X, Y, Z, V) + 4\omega_s(X, Y)\omega_s(Z, V) \right) \right],$$

where $I_s U(X, Y) = -U(X, I_s Y)$ and \oslash is the Kulkarni-Nomizu product of (0,2) tensors, for example,

$$(\omega_s \oslash U)(X, Y, Z, V) := \omega_s(X, Z)U(Y, V) + \omega_s(Y, V)U(X, Z) - \omega_s(Y, Z)U(X, V) - \omega_s(X, V)U(Y, Z).$$

The main result from [23] can be stated as follows

Theorem 2.4. [23] *A qc structure on a $(4n+3)$ -dimensional smooth manifold is locally qc conformal to the standard flat qc structure on the quaternionic Heisenberg group $G(\mathbb{H})$ if and only if the qc conformal curvature vanishes, $W^{qc} = 0$. In this case, we call the qc structure a qc conformally flat structure.*

A qc conformally flat structure is also locally qc conformal to the standard 3-Sasaki sphere due to the local qc conformal equivalence of the standard 3-Sasakian structure on the $4n+3$ -dimensional sphere and the quaternionic Heisenberg group [20, 23].

3. EXPLICIT EXAMPLES OF QUATERNIONIC CONTACT STRUCTURES

In this section we give explicit examples of qc structures in dimension seven satisfying the compatibility conditions (2.1). The first example has zero torsion and is locally qc conformal to the quaternionic Heisenberg group. The second example has zero torsion while the third is with non-vanishing torsion, and both are not locally qc conformal to the quaternionic Heisenberg group. We remind that the zero torsion qc structures are precisely the qc Einstein structures, cf. Theorem 2.3.

Remark 3.1. *We note explicitly that the vanishing of the torsion endomorphism implies that, locally, the structure is homothetic to a 3-Sasakian structure if the qc scalar curvature is positive. In the seven dimensional examples below the qc scalar curvature is a negative constant. In that respect, as pointed by Charles Boyer, there are no compact invariant with respect to translations 3-Sasakian Lie groups of dimension seven.*

3.1. Example 0: The quaternionic Heisenberg group $G(\mathbb{H})$ - the Biquard-flat qc structure. As a manifold $G(\mathbb{H}) = \mathbb{H}^n \times \text{Im } \mathbb{H}$, while the group multiplication is given by $(q', \omega') = (q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2 \text{Im } q_o \bar{q})$, where $q, q_o \in \mathbb{H}^n$ and $\omega, \omega_o \in \text{Im } \mathbb{H}$. The standard flat qc structure is defined by the left-invariant qc form $\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2} (d\omega - q' \cdot d\bar{q}' + dq' \cdot \bar{q}')$, where \cdot denotes the quaternion multiplication. As a Lie group it can be characterized by the following structure equations. Denote by $e^a, 1 \leq a \leq (4n+3)$ the basis of the left invariant 1-forms, and by e^{ij} the wedge product $e^i \wedge e^j$. The $(4n+3)$ -dimensional quaternionic Heisenberg Lie algebra is the 2-step nilpotent Lie algebra defined by:

$$(3.1) \quad \begin{aligned} de^a &= 0, \quad 1 \leq a \leq 4n, \\ d\eta_1 &= de^{4n+1} = 2(e^{12} + e^{34} + \dots + e^{(4n-3)(4n-2)} + e^{(4n-1)4n}) = 2\omega_1, \\ d\eta_2 &= de^{4n+2} = 2(e^{13} + e^{42} + \dots + e^{(4n-3)(4n-1)} + e^{4n(4n-2)}) = 2\omega_2, \\ d\eta_3 &= de^{4n+3} = 2(e^{14} + e^{23} + \dots + e^{(4n-3)4n} + e^{(4n-2)(4n-1)}) = 2\omega_3. \end{aligned}$$

The Biquard connection coincides with the flat left-invariant connection on $\mathbf{G}(\mathbb{H})$. This flat qc structure on the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ is (locally) the unique qc structure with flat Biquard connection [20, 24]. By a rotation of the 1-forms defining the horizontal space of $\mathbf{G}(\mathbb{H})$ we obtain an equivalent qc-structure (with the same Biquard connection). It is possible to introduce a different not two step nilpotent group structure on $\mathbb{H}^n \times \text{Im } \mathbb{H}$ with respect to which the rotated forms are left invariant (but not parallel!). Following is an explicit description of this construction in dimension seven.

Consider the seven dimensional quaternionic Heisenberg group. Since e^4 is closed we can write $e^4 = dx_4$, where x_4 is a global function on the manifold $\mathbb{H} \times \text{Im } \mathbb{H}$. Now we can use this function to define a non-left-invariant qc structure on this manifold as follows. For each $c \in \mathbb{R}$, let

$$(3.2) \quad \begin{aligned} \gamma^1 &= e^1, & \gamma^2 &= \sin(-cx_4) e^2 + \cos(-cx_4) e^3, & \gamma^3 &= -\cos(-cx_4) e^2 + \sin(-cx_4) e^3, & \gamma^4 &= e^4, \\ \gamma^5 &= \sin(-cx_4) e^5 + \cos(-cx_4) e^6, & \gamma^6 &= -\cos(-cx_4) e^5 + \sin(-cx_4) e^6, & \gamma^7 &= e^7. \end{aligned}$$

A direct calculation shows that for $c \neq 0$ the forms $\{\gamma^l, 1 \leq l \leq 7\}$ define a unique Lie algebra \mathfrak{l}_0 with the following structure equations

$$(3.3) \quad \begin{aligned} d\gamma^1 &= 0, & d\gamma^2 &= -c\gamma^{34}, & d\gamma^3 &= c\gamma^{24}, & d\gamma^4 &= 0, \\ d\gamma^5 &= 2\gamma^{12} + 2\gamma^{34} + c\gamma^{46}, & d\gamma^6 &= 2\gamma^{13} + 2\gamma^{42} - c\gamma^{45}, & d\gamma^7 &= 2\gamma^{14} + 2\gamma^{23}. \end{aligned}$$

In particular, \mathfrak{l}_0 is an indecomposable solvable Lie algebra. Let $e_l, 1 \leq l \leq 7$ be the left invariant vector fields dual to the 1-forms $\gamma^l, 1 \leq l \leq 7$. The (global) flat qc structure on $\mathbb{H} \times \text{Im } \mathbb{H}$ can also be described as follows $\eta_1 = \gamma^5, \eta_2 = \gamma^6, \eta_3 = \gamma^7, H = \text{span}\{\gamma^1, \dots, \gamma^4\}, \omega_1 = \gamma^{12} + \gamma^{34}, \omega_2 = \gamma^{13} + \gamma^{42}, \omega_3 = \gamma^{14} + \gamma^{23}$. It is straightforward to check from (3.3) that the vector fields $\xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7$ satisfy the Duchemin compatibility conditions (2.1) and therefore the Biquard connection exists and ξ_s are the Reeb vector fields.

Let (L_0, η, \mathbb{Q}) be the simply connected connected Lie group with Lie algebra \mathfrak{l}_0 equipped with the left invariant qc structure (η, \mathbb{Q}) defined above. Then, as a consequence of the above construction, the torsion endomorphism and the curvature of the Biquard connection are identically zero but the basis $\gamma_1, \dots, \gamma_7$ is not parallel. The $Sp(1)$ -connection 1-forms in the basis $\gamma^1, \dots, \gamma^7$ are given by $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = c\gamma^4$.

3.2. Example 1: zero torsion qc-flat structure. Denote $\{\tilde{e}^l, 1 \leq l \leq 7\}$ the basis of the left invariant 1-forms and consider the simply connected connected Lie group L_1 with indecomposable Lie algebra \mathfrak{l}_1 defined by the following equations

$$(3.4) \quad \begin{aligned} d\tilde{e}^1 &= 0, & d\tilde{e}^2 &= -\tilde{e}^{12} - 2\tilde{e}^{34} - \frac{1}{2}\tilde{e}^{37} + \frac{1}{2}\tilde{e}^{46}, \\ d\tilde{e}^3 &= -\tilde{e}^{13} + 2\tilde{e}^{24} + \frac{1}{2}\tilde{e}^{27} - \frac{1}{2}\tilde{e}^{45}, & d\tilde{e}^4 &= -\tilde{e}^{14} - 2\tilde{e}^{23} - \frac{1}{2}\tilde{e}^{26} + \frac{1}{2}\tilde{e}^{35}, \\ d\tilde{e}^5 &= 2\tilde{e}^{12} + 2\tilde{e}^{34} - \frac{1}{2}\tilde{e}^{67}, & d\tilde{e}^6 &= 2\tilde{e}^{13} + 2\tilde{e}^{42} + \frac{1}{2}\tilde{e}^{57}, & d\tilde{e}^7 &= 2\tilde{e}^{14} + 2\tilde{e}^{23} - \frac{1}{2}\tilde{e}^{56}. \end{aligned}$$

Let $e_l, 1 \leq l \leq 7$ be the left invariant vector field dual to the 1-forms $\tilde{e}^l, 1 \leq l \leq 7$, respectively. A global qc structure on L_1 is defined by

$$(3.5) \quad \begin{aligned} \eta_1 &= \tilde{e}^5, & \eta_2 &= \tilde{e}^6, & \eta_3 &= \tilde{e}^7, & H &= \text{span}\{\tilde{e}^1, \dots, \tilde{e}^4\}, \\ \omega_1 &= \tilde{e}^{12} + \tilde{e}^{34}, & \omega_2 &= \tilde{e}^{13} + \tilde{e}^{42}, & \omega_3 &= \tilde{e}^{14} + \tilde{e}^{23}. \end{aligned}$$

It is straightforward to check from (3.4) that the vector fields $\xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7$ satisfy the Duchemin compatibility conditions (2.1) and therefore the Biquard connection exists and ξ_s are the Reeb vector fields.

Theorem 3.2. *Let (L_1, η, \mathbb{Q}) be the simply connected Lie group with Lie algebra \mathfrak{l}_1 equipped with the left invariant qc structure (η, \mathbb{Q}) defined above. Then*

- a) *The qc structure is qc Einstein the normalized qc scalar curvature is a negative constant, $S = -\frac{1}{2}$.*
- b) *The qc conformal curvature is zero, $W^{qc} = 0$, and therefore (L_1, η, \mathbb{Q}) is locally qc conformally flat.*

Proof. We compute first the connection 1-forms and the horizontal Ricci forms of the Biquard connection. The Lie algebra structure equations (3.4) together with (2.3), (2.4) and (2.6) imply

$$(3.6) \quad \alpha_s = \left(\frac{1}{4} - \frac{S}{2}\right)\eta_s, \quad \rho_s(X, Y) = \frac{1}{2}d\alpha_s(X, Y) = \left(\frac{1}{4} - \frac{S}{2}\right)\omega_s(X, Y).$$

Equation (3.6) and (2.9) allow us to conclude that the torsion endomorphism is zero and the normalized qc scalar $S = -\frac{1}{2}$. Now, Theorem 2.3 completes the proof of part a).

In view of Theorem 2.4, to prove part b) we have to show $W^{qc} = 0$. Since the torsion of the Biquard connection vanishes and $S = -\frac{1}{2}$, (2.10) takes the form

$$(3.7) \quad W^{qc}(X, Y, Z, V) = R(X, Y, Z, V) - \frac{1}{8} \left[(g \otimes g)(X, Y, Z, V) + \sum_{s=1}^3 \left((\omega_s \otimes \omega_s)(X, Y, Z, V) + 4\omega_s(X, Y)\omega_s(Z, V) \right) \right].$$

The Koszul formula expressing the Levi-Civita connection in terms of the metric in the case of left-invariant vector fields A, B, C on a Lie group reads

$$(3.8) \quad g(\nabla_A^g B, C) = \frac{1}{2} \left[g([A, B], C) - g([B, C], A) + g([C, A], B) \right].$$

By Theorem 2.1 we have the formula

$$(3.9) \quad T(X, Y) = 2 \sum_{s=1}^3 \omega_s(X, Y) \xi_s.$$

Using (3.9), (3.8), (2.2) and the structure equations (3.4) we found the non zero coefficients of the curvature tensor are $R(e_a, e_b, e_a, e_b) = -R(e_a, e_b, e_b, e_a) = 1$, $a, b = 1, \dots, 4$, $a \neq b$. Now (3.7) yields $W^{qc}(e_a, e_b, e_c, e_d) = R(e_a, e_b, e_c, e_d) = 0$, when there are three different indices in a, b, c, d . For the indices repeated in pairs we have

$$\begin{aligned} W^{qc}(e_a, e_b, e_a, e_b) &= R(e_a, e_b, e_a, e_b) - \frac{1}{8}(g \otimes g)(e_a, e_b, e_a, e_b) - \\ &\quad \frac{1}{8} \left[\sum_{s=1}^3 \left((\omega_s \otimes \omega_s)(e_a, e_b, e_a, e_b) + 4\omega_s(e_a, e_b)\omega_s(e_a, e_b) \right) \right] = 1 - \frac{2}{8} - \frac{6}{8} = 0 \end{aligned}$$

Then Theorem 2.4 completes the proof. \square

The Lie algebra of the group L_1 is a semi-direct sum, $l_1 = su(2) \oplus_{\pi} \mathfrak{a}_{4,5}$, of $su(2)$ and the four dimensional solvable Lie algebra $\mathfrak{a}_{4,5}$, [29], given, respectively, by

$$\begin{aligned} su(2) : \quad & df^5 = -\frac{1}{2}f^{67}, \quad df^6 = -\frac{1}{2}f^{75}, \quad df^7 = -\frac{1}{2}f^{56}, \\ \mathfrak{a}_{4,5} : \quad & d\tilde{e}^1 = 0, \quad d\tilde{e}^2 = -\tilde{e}^{12}, \quad d\tilde{e}^3 = -\tilde{e}^{13}, \quad d\tilde{e}^4 = -\tilde{e}^{14}. \end{aligned}$$

The action π of $su(2)$ on $\mathfrak{a}_{4,5}$ is the restriction to $su(2)$ of ad on L_1 , i.e., with the notation $\pi(f_i)\tilde{e}_j \equiv [f_i, \tilde{e}_j]$ the action is

$$\begin{aligned} [f_5, \tilde{e}_3] &= \frac{1}{2}\tilde{e}_4, \quad [f_5, \tilde{e}_4] = -\frac{1}{2}\tilde{e}_3 \\ [f_6, \tilde{e}_4] &= \frac{1}{2}\tilde{e}_2, \quad [f_6, \tilde{e}_2] = -\frac{1}{2}\tilde{e}_4 \\ [f_7, \tilde{e}_2] &= \frac{1}{2}\tilde{e}_3, \quad [f_7, \tilde{e}_3] = -\frac{1}{2}\tilde{e}_2, \end{aligned}$$

where f_i and \tilde{e}_j are the dual vectors. This decomposition can be seen easier in the basis

$$f^1 = e^1, \quad f^2 = e^2, \quad f^3 = e^3, \quad f^4 = e^4, \quad f^5 = 2e^2 + e^5, \quad f^6 = 2e^3 + e^6, \quad f^7 = 2e^4 + e^7,$$

which satisfy the structure equations

$$\begin{aligned} df^1 &= 0, \quad df^2 = -f^{12} - \frac{1}{2}f^{37} + \frac{1}{2}f^{46}, \quad df^3 = -f^{13} + \frac{1}{2}f^{27} - \frac{1}{2}f^{45}, \quad df^4 = -f^{14} - \frac{1}{2}f^{26} + \frac{1}{2}f^{35} \\ df^5 &= -\frac{1}{2}f^{67}, \quad df^6 = -\frac{1}{2}f^{75}, \quad df^7 = -\frac{1}{2}f^{56}. \end{aligned}$$

3.3. Example 2: zero torsion qc-non-flat structure. Consider the simply connected connected Lie group L_2 with Lie algebra defined by the equations:

$$\begin{aligned} (3.10) \quad de^1 &= 0, \quad de^2 = -e^{12} + e^{34}, \quad de^3 = -\frac{1}{2}e^{13}, \quad de^4 = -\frac{1}{2}e^{14}, \\ de^5 &= 2e^{12} + 2e^{34} + e^{37} - e^{46} + \frac{1}{4}e^{67}, \quad de^6 = 2e^{13} - 2e^{24} - \frac{1}{2}e^{27} + e^{45} - \frac{1}{4}e^{57}, \\ de^7 &= 2e^{14} + 2e^{23} + \frac{1}{2}e^{26} - e^{35} + \frac{1}{4}e^{56}. \end{aligned}$$

A global qc structure on L_2 is defined by (3.5). It is easy to check from (3.10) that the triple $\{\xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7\}$ form the Reeb vector fields satisfying (2.1) and therefore the Biquard connection do exists.

Theorem 3.3. *Let (L_2, η, \mathbb{Q}) be the simply connected Lie group with Lie algebra \mathfrak{l}_2 equipped with the left invariant qc structure (η, \mathbb{Q}) defined above. Then:*

- a) *The qc structure is qc Einstein and the normalized qc scalar curvature is a negative constant, $S = -\frac{1}{4}$.*
- b) *The qc conformal curvature $W^{qc} \neq 0$ and therefore (L_2, η, \mathbb{Q}) is not locally qc conformally flat.*

Proof. We compute the $sp(1)$ -connection 1-forms and the horizontal Ricci forms of the Biquard connection. The Lie algebra structure equations (3.10) together with (2.3), (2.4) and (2.6) imply

$$\begin{aligned} (3.11) \quad \alpha_1 &= -\frac{1}{2}e^2 - \left(\frac{1}{8} + \frac{S}{2}\right)\eta_1, \quad \alpha_2 = -e^3 - \left(\frac{1}{8} + \frac{S}{2}\right)\eta_2, \quad \alpha_3 = -e^4 - \left(\frac{1}{8} + \frac{S}{2}\right)\eta_3, \\ \rho_s(X, Y) &= \left(\frac{1}{8} - \frac{S}{2}\right)\omega_s(X, Y). \end{aligned}$$

Compare (3.11) with (2.9) to conclude that the torsion is zero and the normalized qc scalar $S = -\frac{1}{4}$. Theorem 2.3 completes the proof of part a).

In view of Theorem 2.4, we have to show $W^{qc}(e_1, e_2, e_3, e_4) = R(e_1, e_2, e_3, e_4) \neq 0$. Indeed, using (3.9), (3.8), (2.2) and the structure equations (3.10) we found $R(e_1, e_2, e_3, e_4) = -\frac{1}{2} \neq 0$. \square

The Lie algebra \mathfrak{l}_2 of the group L_2 is a direct sum of $su(2)$ and the four dimensional solvable algebra $\mathfrak{a}_{4,8}$, [29], given respectively by

$$\begin{aligned} su(2) : \quad d\tilde{e}^5 &= \frac{1}{4}\tilde{e}^{67}, \quad d\tilde{e}^6 = \frac{1}{4}\tilde{e}^{75}, \quad d\tilde{e}^7 = \frac{1}{4}\tilde{e}^{56} \\ \mathfrak{a}_{4,8} : \quad d\tilde{e}^1 &= 0, \quad d\tilde{e}^2 = -\tilde{e}^{12} + \tilde{e}^{34}, \quad d\tilde{e}^3 = -\frac{1}{2}\tilde{e}^{13}, \quad d\tilde{e}^4 = -\frac{1}{2}\tilde{e}^{14}. \end{aligned}$$

This decomposition can be seen by letting

$$e^5 = -2\tilde{e}^2 + \tilde{e}^5, \quad e^6 = -4\tilde{e}^3 + \tilde{e}^6, \quad e^7 = -4\tilde{e}^4 + \tilde{e}^7, \quad e^m = \tilde{e}^m, \quad m = 1, 2, 3, 4.$$

3.4. Example 3: non-zero torsion qc-non-flat structure. Consider the solvable indecomposable Lie algebra \mathfrak{l}_3 defined by the equations

$$\begin{aligned}
 (3.12) \quad de^1 &= -\frac{3}{2}e^{13} + \frac{3}{2}e^{24} - \frac{3}{4}e^{25} + \frac{1}{4}e^{36} - \frac{1}{4}e^{47} + \frac{1}{8}e^{57}, \\
 de^2 &= -\frac{3}{2}e^{14} - \frac{3}{2}e^{23} + \frac{3}{4}e^{15} + \frac{1}{4}e^{37} + \frac{1}{4}e^{46} - \frac{1}{8}e^{56}, \\
 de^3 &= 0, \quad de^4 = e^{12} + e^{34} + \frac{1}{2}e^{17} - \frac{1}{2}e^{26} + \frac{1}{4}e^{67}, \\
 de^5 &= 2e^{12} + 2e^{34} + e^{17} - e^{26} + \frac{1}{2}e^{67}, \\
 de^6 &= 2e^{13} + 2e^{42} + e^{25}, \quad de^7 = 2e^{14} + 2e^{23} - e^{15},
 \end{aligned}$$

and let $e_l, 1 \leq l \leq 7$ be the left invariant vector field dual to the 1-forms $e^l, 1 \leq l \leq 7$. We define a global qc structure on the corresponding Lie group L_3 by (3.5). It follows from (3.12) that the triple $\{\xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7\}$ form the Reeb vector fields satisfying (2.1) and therefore the Biquard connection exists.

Theorem 3.4. *Let (L_3, η, \mathbb{Q}) be the simply connected connected Lie group with Lie algebra \mathfrak{l}_3 equipped with the left invariant qc structure (η, \mathbb{Q}) defined by (3.5). Then*

- a) *The torsion endomorphism of the Biquard connection is not zero and the normalized qc scalar curvature is negative, $S = -1$.*
- b) *The qc conformal curvature $W^{qc} \neq 0$, and therefore (L_3, η, \mathbb{Q}) is not locally qc conformally flat.*

Proof. The structure equations (3.12) together with (2.3), (2.4) imply

$$(3.13) \quad \alpha_1 = \left(\frac{1}{4} - \frac{S}{2}\right)\eta_1, \quad \alpha_2 = -e^1 - \left(\frac{1}{4} + \frac{S}{2}\right)\eta_2, \quad \alpha_3 = -e^2 - \left(\frac{1}{4} + \frac{S}{2}\right)\eta_3.$$

Now, (3.13), (3.12) and (2.6) yield

$$\begin{aligned}
 (3.14) \quad \rho_1(X, Y) &= \frac{1}{2} \left[\left(\frac{1}{2} - S \right) (e^{12} + e^{34}) + e^{12} \right] (X, Y) = \frac{1}{4} (e^{12} - e^{34}) (X, Y) + \frac{1}{2} (1 - S) \omega_1(X, Y), \\
 \rho_2(X, Y) &= \frac{1}{2} \left[\frac{3}{2} (e^{13} - e^{24}) (X, Y) - \left(\frac{1}{2} + S \right) (e^{13} - e^{24}) (X, Y) \right] + \frac{1}{2} (1 - S) \omega_2(X, Y), \\
 \rho_3(X, Y) &= \frac{1}{2} \left[\frac{3}{2} (e^{14} + e^{23}) (X, Y) - \left(\frac{1}{2} + S \right) (e^{14} + e^{23}) (X, Y) \right] + \frac{1}{2} (1 - S) \omega_3(X, Y).
 \end{aligned}$$

Comparing (3.14) with (2.9) we conclude

$$\begin{aligned}
 (3.15) \quad T^0(X, I_1 Y) - T^0(I_1 X, Y) &= \frac{1}{2} (e^{12} - e^{34}) (X, Y), \quad S = -1, \\
 T^0(X, I_2 Y) - T^0(I_2 X, Y) &= 0, \quad T^0(X, I_3 Y) - T^0(I_3 X, Y) = 0.
 \end{aligned}$$

Now we compute W^{qc} . Denote $\psi = -\frac{1}{4}(e^{12} - e^{34})$ and compare (3.15) with (2.7) and (2.8) to get

$$(3.16) \quad T^0(X, Y) = \psi(X, I_1 Y), \quad g(T(\xi_s, X), Y) = -\frac{1}{4}(\psi(I_s X, I_1 Y) + \psi(X, I_1 I_s Y)).$$

Using $U = 0$ and (2.7) we conclude from (2.10) that $W^{qc}(e_1, e_2, e_3, e_4) = R(e_1, e_2, e_3, e_4)$ since other terms on the right hand side of (2.10) vanish on the quadruple $\{e_1, e_2 = -I_1 e_1, e_3 = -I_2 e_1, e_4 = -I_3 e_1\}$. Using (2.2), (3.8), (3.9), (3.12) and (3.16), we obtain $R(e_1, e_2, e_3, e_4) = -\frac{1}{2} \neq 0$. \square

4. $Sp(n)Sp(1)$ -HYPO STRUCTURES AND QUATERNIONIC KÄHLER MANIFOLDS

Guided by Examples 0–3 in the last Section we relax the definition of a qc structure by removing the “contact condition” $d\eta_{s|_H} = 2\omega_s$. In this way we come to an $Sp(n)Sp(1)$ structure (almost 3-contact structure see [26]). The goal is to obtain a geometric structure which may induce a quaternionic Kähler metric on a product of the given manifold with (an interval of) the real line.

Definition 4.1. An $Sp(n)Sp(1)$ structure on a $(4n+3)$ -dimensional Riemannian manifold (M, g) is a codimension three distribution H such that

- i) H has an $Sp(n)Sp(1)$ structure, that is, it is equipped with a Riemannian metric g and a rank-three bundle \mathbb{Q} consisting of $(1,1)$ -tensors on H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1 I_2 = -I_2 I_1 = I_3$, $I_1 I_2 I_3 = -\text{id}_H$ which are hermitian compatible with the metric $g(I_s \cdot, I_s \cdot) = g(\cdot, \cdot)$, i.e. H has an almost quaternion hermitian structure.

- ii) H is locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 .

The local fundamental 2-forms are defined on H as usual by $\omega_s(X, Y) = g(I_s X, Y)$.

If the first Pontrjagin class of M vanishes then the 1-forms η_s as well as the fundamental 2-forms ω_s are globally defined [2].

Definition 4.2. We define a global $Sp(n)Sp(1)$ -invariant 4-form of an $Sp(n)Sp(1)$ structure (M, g, \mathbb{Q}) on a $(4n+3)$ -dimensional manifold M by the formula

$$(4.1) \quad \Omega_{\mathbb{Q}} = \omega_1^2 + \omega_2^2 + \omega_3^2 + 2\omega_1 \wedge \eta_2 \wedge \eta_3 + 2\omega_2 \wedge \eta_3 \wedge \eta_1 + 2\omega_3 \wedge \eta_1 \wedge \eta_2.$$

Let M^{4n+4} be a $(4n+4)$ -dimensional manifold equipped with an $Sp(n+1)Sp(1)$ structure, i.e. $(M^{4n+4}, g, J_1, J_2, J_3)$ is an almost quaternion hermitian manifold with local Kähler forms $F_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$. The fundamental 4-form

$$(4.2) \quad \Phi = F_1 \wedge F_1 + F_2 \wedge F_2 + F_3 \wedge F_3$$

is globally defined and encodes fundamental properties of the structure. If the holonomy of the Levi-Civita connection is contained in $Sp(n+1)Sp(1)$ then the manifold is a quaternionic Kähler manifold which is consequently an Einstein manifold. Equivalent conditions are either that the fundamental 4-form Φ is parallel with respect to the Levi-Civita connection or Φ is a closed form and

$$(4.3) \quad dF_s = 0 \mod \{F_i, F_j, F_k\}$$

[34]. The latter is equivalent to the fact that the fundamental 4-form is closed ($d\Phi = 0$) provided the dimension is strictly bigger than eight ([33, 34, 32]) with a counter-example in dimension eight constructed by Salamon in [32].

Let $f : N^{4n+3} \rightarrow M^{4n+4}$ be an oriented hypersurface of M^{4n+4} and denote by \mathbb{N} the unit normal vector field. Then an $Sp(n+1)Sp(1)$ structure on M induces an $Sp(n)Sp(1)$ structure on N^{4n+3} locally given by (η_s, ω_s) defined by the equalities $\eta_s = \mathbb{N} \lrcorner F_s$, $\omega_i = f^* F_i - \eta_j \wedge \eta_k$. The fundamental four form Φ on M restricts to the four form $\Omega_{\mathbb{Q}}$ on N , $\Omega_{\mathbb{Q}} = f^* \Phi = (f^* F_1)^2 + (f^* F_2)^2 + (f^* F_3)^2$.

Suppose that (M^{4n+4}, g) has holonomy contained in $Sp(n+1)Sp(1)$. Then $d\Phi = 0$ implies that the $Sp(n)Sp(1)$ structure induced on N^{4n+3} satisfies the equation

$$(4.4) \quad d\Omega_{\mathbb{Q}} = 0,$$

since d commutes with f^* , $df^* = f^* d$.

Definition 4.3. An $Sp(n)Sp(1)$ structure (M, g, \mathbb{Q}) on a $(4n+3)$ -dimensional manifold is called $Sp(n)Sp(1)$ -hypo if the 4-form $\Omega_{\mathbb{Q}}$ is closed, $d\Omega_{\mathbb{Q}} = 0$.

Hence, any oriented hypersurface N^{4n+3} of a quaternionic Kähler M^{4n+4} is naturally endowed with an $Sp(n)Sp(1)$ -hypo structure.

Vice versa, a $(4n+3)$ -manifold N^{4n+3} with an $Sp(n)Sp(1)$ structure (η_s, ω_s) induces an $Sp(n+1)Sp(1)$ structure (F_s) on $N^{4n+3} \times \mathbb{R}$ defined by

$$(4.5) \quad F_i = \omega_i + \eta_j \wedge \eta_k - \eta_i \wedge dt,$$

where t is a coordinate on \mathbb{R} .

Consider the family of $Sp(n)Sp(1)$ structures $(\eta_s(t), \omega_s(t))$ on N^{4n+3} depending on a real parameter $t \in \mathbb{R}$, and the corresponding $Sp(n+1)Sp(1)$ structures $F_s(t)$ on $N^{4n+3} \times \mathbb{R}$.

Proposition 4.4. *An $Sp(n)Sp(1)$ structure (η_s, ω_s) on N^{4n+3} can be lifted to an almost quaternionic hermitian structure $(F_s(t))$ with a closed four form on $N^{4n+3} \times \mathbb{R}$ defined by (4.5) if and only if it is an $Sp(n)Sp(1)$ -hypo structure which generates a 1-parameter family of $Sp(n)Sp(1)$ -hypo structures $(\eta_s(t), \omega_s(t))$ satisfying the following evolution $Sp(n)Sp(1)$ -hypo equations*

$$(4.6) \quad \partial_t \Omega_{\mathbb{Q}}(t) = d \left[6\eta_1(t) \wedge \eta_2(t) \wedge \eta_3(t) + 2\omega_1(t) \wedge \eta_1(t) + 2\omega_2(t) \wedge \eta_2(t) + 2\omega_3(t) \wedge \eta_3(t) \right],$$

where d is the exterior derivative on N .

If $n \geq 2$, then the almost quaternionic hermitian structure $(F_s(t))$ with a closed four form on $N^{4n+3} \times \mathbb{R}$ defined by (4.5) is quaternionic Kähler.

Proof. If we apply (4.5) to (4.2) and then take the exterior derivative in the obtained equation we see that the equality $d\Phi = 0$ holds precisely when (4.4) and (4.6) are fulfilled.

It remains to show that the equations (4.6) imply that (4.4) holds for each t . Using (4.6), we calculate

$$\partial_t d\Omega_{\mathbb{Q}} = d^2 \left[6\eta_1(t) \wedge \eta_2(t) \wedge \eta_3(t) + 2\omega_1(t) \wedge \eta_1(t) + 2\omega_2(t) \wedge \eta_2(t) + 2\omega_3(t) \wedge \eta_3(t) \right] = 0.$$

Hence, the equalities (4.4) are independent of t and therefore valid for all t since they hold for $t = 0$. \square

It is interesting to know whether the converse of Proposition 4.4 holds, i.e., is it true that any $Sp(n)Sp(1)$ -hypo structure on N^{4n+3} , $n > 2$, can be lifted to a quaternionic Kähler structure on $N^{4n+3} \times \mathbb{R}$?

Solutions to (4.4) are given in the case of 3-Sasakian manifolds in [37]. We construct explicit examples relying on the properties of the qc structures for which we solve the evolution equations (4.6).

4.1. Quaternionic contact and $Sp(n)Sp(1)$ -hypo structures. We show the conditions under which a qc structure is an $Sp(n)Sp(1)$ -hypo. For $n = 1$, we prove that it happens exactly when the vertical distribution is integrable, while for $n > 1$ we have that a qc structure is $Sp(n)Sp(1)$ -hypo if and only if it is qc Einstein. To this end we recall that the structure equations of a qc structure, discovered in [24], are

$$(4.7) \quad 2\omega_i = d\eta_i + \eta_j \wedge \alpha_k - \eta_k \wedge \alpha_j + S\eta_j \wedge \eta_k,$$

and a 3-Sasakian qc structure is characterized by the structure equations $2\omega_i = d\eta_i - 2\eta_j \wedge \eta_k$.

The $Sp(n)Sp(1)$ -invariant horizontal 4-form, called the fundamental 4-form, is defined in [24] by

$$(4.8) \quad \Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

If the dimension of the qc manifold is bigger than seven it turns out that qc Einstein condition is equivalent the fundamental 4-form Ω being closed, see [24].

Proposition 4.5. *A qc structure is $Sp(n)Sp(1)$ -hypo structure if and only if its fundamental four form is closed, $d\Omega = 0$.*

Proof. Comparing the definitions of $\Omega_{\mathbb{Q}}$ and Ω we see that it is sufficient to show that the four form

$$\omega_1 \wedge \eta_2 \wedge \eta_3 + \omega_2 \wedge \eta_3 \wedge \eta_1 + \omega_3 \wedge \eta_1 \wedge \eta_2$$

is closed. The structure equations (4.7) imply [24]

$$(4.9) \quad d\omega_i = \omega_j \wedge (\alpha_k + S\eta_k) - \omega_k \wedge (\alpha_j + S\eta_j) - \rho_k \wedge \eta_j + \rho_j \wedge \eta_k + \frac{1}{2}dS \wedge \eta_j \wedge \eta_k.$$

Using (4.7) and (4.9) we obtain the validity of the next

Lemma 4.6. *For any qc structure we have*

$$d(\omega_1 \wedge \eta_2 \wedge \eta_3 + \omega_2 \wedge \eta_3 \wedge \eta_1 + \omega_3 \wedge \eta_1 \wedge \eta_2) = 0.$$

Now, Lemma (4.6) implies $d\Omega_{\mathbb{Q}} = 0$ precisely when $d\Omega = 0$. \square

The main theorem of [24] asserts that a qc structure on a qc manifold of dimension strictly bigger than seven has closed fundamental four form exactly when it has zero torsion endomorphism of the Biquard connection. In dimension seven we have the following result.

Theorem 4.7. *For a qc structure in dimension seven the next conditions are equivalent.*

- a) *The fundamental four form is closed, $d\Omega = 0$.*
- b) *The vertical distribution is integrable.*
- c) *The qc structure is $Sp(1)Sp(1)$ -hypo structure.*

Proof. In the case of dimension seven we have the identities $\omega_s \wedge \omega_p = \delta_{sq} \text{vol}_H$. Therefore, $\Omega = 3\omega_1 \wedge \omega_1$. Applying (4.9) we get

$$(4.10) \quad d\Omega = 6d\omega_1 \wedge \omega_1 = 6\omega_1 \wedge \rho_2 \wedge \eta_3 - 6\omega_1 \wedge \rho_3 \wedge \eta_2 + 3dS \wedge \omega_1 \wedge \eta_2 \wedge \eta_3.$$

For a 1-form β we use the convention $I_s\beta(X) = -\beta(I_sX)$. We obtain from (4.10) that the nonzero parts of $d\Omega$ are given by

$$(4.11) \quad d\Omega(X, Y, Z, \xi_1, \xi_2) = 6\omega_1 \wedge (\xi_1 \lrcorner \rho_3)(X, Y, Z); \quad d\Omega(X, Y, Z, \xi_1, \xi_3) = 6\omega_1 \wedge (\xi_1 \lrcorner \rho_2)(X, Y, Z);$$

$$(4.12) \quad d\Omega(X, Y, Z, \xi_2, \xi_3) = -3\omega_1 \wedge \left[2(\xi_2 \lrcorner \rho_2) + 2(\xi_3 \lrcorner \rho_3) - dS \right](X, Y, Z) = 6\omega_1 \wedge I_3(\xi_2 \lrcorner \rho_1)(X, Y, Z);$$

$$(4.13) \quad d\Omega(X, Y, \xi_1, \xi_2, \xi_3) = 3\omega_1(X, Y) \left[2\rho_2(\xi_1, \xi_2) + 2\rho_3(\xi_1, \xi_3) + dS(\xi_1) \right] = 0;$$

$$(4.14) \quad d\Omega(X, Y, Z, U, \xi_3) = 6\omega_1 \wedge \rho_2(X, Y, Z, U) = g(\omega_1, \rho_2) \text{vol}_H = 0,$$

where we apply the last equation in (2.9) to obtain the last equality in (4.12). The second equality in (4.13) is precisely the second formula of the third line of (2.9). The last equality in (4.14) follows from the first formula in (2.9) which says that the horizontal two form ρ_2 is of type (1,1) with respect to I_2 and therefore it is orthogonal to the 2-form ω_1 which is of type (2,0)+(0,2) with respect to I_2 .

Assume that the fundamental four form is closed. Then (4.11) and (4.12) imply that the 1-forms $(\xi_s \lrcorner \rho_t)|_H$ vanish for $s \neq t$ which is equivalent the vertical distribution to be integrable due to the third line of (2.9).

For the converse, let the vertical distribution be integrable, i.e. $(i_{\xi_t} \rho_s)|_H = 0$ for $s \neq t$. Then the last formula in (2.9) yields $(i_{\xi_s} \rho_s)|_H = \frac{1}{4}dS|_H$. Substitute the latter into (4.12) to obtain the vanishing of this term which combined with (4.11), (4.13) and (4.14) yields $d\Omega = 0$. This proves $a) \Leftrightarrow b)$. Finally, Proposition 4.5 completes the proof. \square

Combining Proposition 4.5 with the main theorem of [24] we obtain the following Corollary.

Corollary 4.8. *a) A qc structure on a $(4n+3)$ -dimensional manifold is $Sp(n)Sp(1)$ -hypo if and only if it is qc Einstein provided $n > 1$.*

b) In dimension seven, a qc Einstein structure with constant qc scalar curvature is $Sp(1)Sp(1)$ -hypo structure.

In both cases we have the structure equations

$$(4.15) \quad d\omega_i = \omega_j \wedge \alpha_k - \omega_k \wedge \alpha_j.$$

Proof. If $n > 1$ and $T^0 = U = 0$, then Theorem 2.3 implies

$$(4.16) \quad \rho_l(X, Y) = -S\omega_l(X, Y), \quad \rho_l(\xi_m, X) = 0, \quad \rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j) = 0,$$

since S is constant and the vertical distribution is integrable [20]. In dimension seven the second equality of (4.16) is a consequence of Theorem 3.1 in [23] which expresses $\rho_l(\xi_m, X)$ in terms of the covariant derivatives of the torsion and the differential of the qc scalar curvature. Applying (4.16) into (4.9) we get (4.15). \square

4.2. Construction of quaternionic Kähler structures using qc structures. In this section we construct explicit quaternionic Kähler metrics on the product of a qc manifold with a real interval.

Theorem 4.9. *Let (M, g, \mathbb{Q}) be a smooth qc Einstein manifold of dimension $4n + 3$ and, in dimension seven, with constant normalized qc scalar curvature S . For a suitable constant a , the manifold $M \times \mathbb{R}$ has a quaternionic Kähler structure given by the following metric and fundamental 4-form*

$$(4.17) \quad g = u g_H + \left(\frac{1}{2} S u + a u^2 \right) (\eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{1}{2(S u + 2a u^2)} (du)^2, \quad S u + 2a u^2 > 0,$$

$$\Phi = F_1 \wedge F_1 + F_2 \wedge F_2 + F_3 \wedge F_3,$$

where locally

$$(4.18) \quad F_i(u) = u\omega_i + (au^2 + \frac{1}{2}Su)\eta_j \wedge \eta_k - \frac{1}{2}\eta_i \wedge du.$$

The Ricci tensor of the quaternionic Kähler metric is $Ric = -4(n+3)ag$.

Proof. Let h and f be some functions of the unknown t . Consider the 2-forms defined by

$$(4.19) \quad F_i(t) = f(t)\omega_i + h^2(t)\eta_j \wedge \eta_k - h(t)\eta_i \wedge dt.$$

Let Φ be given with the second equation in (4.17). A direct calculation applying (4.7) and (4.15) shows that $(\Sigma_{(ijk)})$ means the cyclic sum

$$d\Phi = \Sigma_{(ijk)} \left[((f^2)' - 4fh)\omega_i \wedge \omega_i \wedge dt + \left(2(fh^2)' + 2Sfh - 12h^3 \right) \omega_i \wedge \eta_j \wedge \eta_k \wedge dt \right].$$

Thus, if we take $h = \frac{1}{2}f'$ we come to

$$d\Phi = f'\Sigma_{(ijk)}(-f'^2 + ff'' + Sf)\omega_i \wedge \eta_j \wedge \eta_k \wedge dt,$$

which shows that Φ is closed exactly when

$$(4.20) \quad ff'' - f'^2 + Sf = 0, \quad h = \frac{1}{2}f'.$$

With the help of the substitution $v = -\ln f$ we see that $\left(\frac{dv}{dt}\right)^2 = 2Se^v + 4a$ for any constant a . This shows that $\left(\frac{dt}{df}\right)^2 = \left(\frac{dt}{dv}\right)^2 \left(\frac{dv}{df}\right)^2 = \frac{1}{2(Sf+2af^2)} > 0$ and $h^2 = \frac{1}{2}Sf + af^2$. Renaming f to u gives the quaternionic structure in the local form (4.18) and the metric in (4.17).

In dimension seven, in order to see that $\langle F_1, F_2, F_3 \rangle$ is a differential ideal when equations (4.20) hold, we need to compute the differentials dF_i . Using (4.7) and (4.15) we obtain taking the exterior derivative of (4.19) that

$$(4.21) \quad dF_i = \left(\alpha_k + \frac{2h^2}{f}\eta_k \right) \wedge F_j - \left(\alpha_j + \frac{2h^2}{f}\eta_j \right) \wedge F_k \\ + (f' - 2h)\omega_i \wedge dt + (2hh' + hS - \frac{4h^3}{4})\eta_j \wedge \eta_k \wedge dt.$$

An easy application of (4.20) annihilates the second line of (4.21) which proves that the defined structure is quaternionic Kähler due to Swann's theorem [33] recalling that we have also (4.20), i.e., Φ is a closed form. \square

Using the above theorem we obtain explicit quaternionic Kähler structures based on examples of qc structures with vanishing torsion. We turn to their description in the following subsection.

4.3. Quaternionic Kähler metrics based on qc Einstein structure with zero qc scalar curvature.

When the qc scalar curvature vanishes, $S = 0$, we let $a = b^2$, $u = e^{2b\sigma}$ in the above Theorem to obtain the next Corollary.

Corollary 4.10. *Let (M, g, \mathbb{Q}) be a smooth qc Einstein manifold of dimension $4n+3$ with vanishing qc scalar curvature. For any non-zero constant b , the manifold $M \times \mathbb{R}$ has a quaternionic Kähler structure given by the following metric and fundamental 4-form*

$$(4.22) \quad g = e^{2b\sigma}g_H + b^2e^{4b\sigma}(\eta_1^2 + \eta_2^2 + \eta_3^2) + (d\sigma)^2, \\ \Phi = F_1 \wedge F_1 + F_2 \wedge F_2 + F_3 \wedge F_3,$$

where locally

$$(4.23) \quad F_i = e^{2b\sigma}\omega_i + b^2e^{4b\sigma}\eta_j \wedge \eta_k - be^{2b\sigma}\eta_i \wedge d\sigma.$$

In particular, the quaternionic Kähler metric on $M \times \mathbb{R}$ is complete if the metric on M is complete.

Proof. The completeness follows similarly to the case of a warped product with strictly positive warping function, see [30] \square

4.3.1. *Quaternionic Kähler metrics from the quaternionic Heisenberg group.* Consider the $(4n+3)$ -dimensional quaternionic Heisenberg group \mathbb{G}^n , viewed as a qc structure. According to (4.22), the metric

$$(4.24) \quad g = e^{2b\sigma} ((e^1)^2 + \cdots + (e^{4n})^2) + b^2 e^{4b\sigma} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + d\sigma^2$$

is a complete quaternionic Kähler metric in dimensions $4n+4$ with $n \geq 1$. The Einstein constant is negative and equal to $-4(n+3)b^2$. This complete Einstein metric has been found in dimension eight on a solvable Lie group as an Einstein metric starting with a T^3 bundle over T^4 in [17, equation (148)]. Thus, the Einstein metric in dimension eight discovered in [17] is in fact a quaternionic Kähler metric. Similarly to the eight dimensional case, the metric (4.24) is a left invariant metric on a $4n+4$ dimensional Lie group. In order to see this we use the structure equation (3.1) giving $de^i = 0$, hence the one forms $\tilde{e}^i = e^{b\sigma} e^i$, $\tilde{\eta}_i = e^{2b\sigma} \eta_i$ and $d\sigma$ define a $4n+4$ -dimensional Lie algebra

$$d\tilde{e}^i = -b\tilde{e}^i \wedge d\sigma, \quad d\tilde{\eta}_i = 2\tilde{\omega}_i - 2b\tilde{\eta}_i \wedge d\sigma,$$

where $\tilde{\omega}_i$ is obtained from ω_i by replacing e^i with \tilde{e}^i .

The explicit description of the quaternionic Kähler metric can be obtained from the qc structure of the quaternionic Heisenberg group (using the form of [20]),

$$(4.25) \quad \begin{aligned} \eta_1 &= \frac{1}{2} dx - x^\alpha dt^\alpha + t^\alpha dx^\alpha - z^\alpha dy^\alpha + y^\alpha dz^\alpha, \\ \eta_2 &= \frac{1}{2} dy - y^\alpha dt^\alpha + z^\alpha dx^\alpha + t^\alpha dy^\alpha - x^\alpha dz^\alpha, \\ \eta_3 &= \frac{1}{2} dz - z^\alpha dt^\alpha - y^\alpha dx^\alpha + x^\alpha dy^\alpha + t^\alpha dz^\alpha, \end{aligned}$$

with summation over $\alpha = 1, \dots, n$. The horizontal forms e^i are dt^α , dx^α , dy^α and dz^α .

4.4. Quaternionic Kähler metrics based on a qc Einstein structure with negative qc scalar curvature. Let us consider a qc Einstein structure with a negative qc scalar curvature. Accordingly, we let $S = -k^2 < 0$ and also replace a in Theorem 4.9 with $\frac{1}{2}b^2k^2 > 0$. With this notation and in terms of the coordinate σ , $u = \frac{1}{2b^2}(1 + \cosh \sigma)$, we have $Su + 2au^2 = k^2(b^2u^2 - u) = \frac{k^2}{4b^2} \sinh^2 \sigma$ and the metric (4.17) takes the form

$$(4.26) \quad g = \frac{1}{2b^2}(1 + \cosh \sigma) g_H + \frac{k^2}{8b^2} \sinh^2 \sigma (\eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{1}{2k^2b^2} d\sigma^2,$$

defined on $\hat{M} = M \times \mathbb{R}$. The Ricci tensor of g is then $\text{Ric} = -2(n+3)b^2k^2g$. Notice that the above metric degenerates only when $\sigma = 0$, but defines a sub-Riemannian metric on the distribution \hat{H} spanned by H and $\frac{\partial}{\partial \sigma}$ since $(1 + \cosh \sigma) \geq 2$ and \hat{H} generates the whole tangent space. From the formula for g it is apparent that if v is any tangent vector to $M \times \mathbb{R}$ then $g(v, v) \geq |d\sigma(v)|^2$ and $g(v, v) \geq g_H(v, v)$. Furthermore, if γ is a horizontal curve on $M \times \mathbb{R}$, i.e., $\dot{\gamma} \in \hat{H}$, then its projection on M is also horizontal curve.

Thus, the lengths of the projections on \mathbb{R} and M of any horizontal curve on $M \times \mathbb{R}$ are less than the length of the horizontal curve on \hat{M} . Let $(p_n, \sigma_n) \in \hat{M}$ be a Cauchy sequence in the sub-Riemannian metric g . From the argument so far we see that the sequences $\sigma_n \in \mathbb{R}$ and $p_n \in M$ are Cauchy sequences in the metric $d\sigma^2$ and the sub-Riemannian metric g_H on M , respectively. It follows that if the sub-Riemannian metric g_H is complete then the sub-Riemannian metric g on \hat{M} is complete. As far as completeness of sub-Riemannian metrics is concerned, it is useful to have in mind the result of [35, Theorem 7.4] according to which if a sub-Riemannian metric has a Riemannian contraction which is complete, then the sub-Riemannian metric is also complete. In particular, if $g_H + \eta_1^2 + \eta_2^2 + \eta_3^2$ is a complete metric on M , then g_H defines a complete sub-Riemannian metric on M and g defines a complete sub-Riemannian metric on $M \times \mathbb{R}$.

Next, we give some examples.

4.4.1. *Explicit quaternionic Kähler metrics from the zero-torsion qc-flat qc structure on \mathbf{I}_1 .* As example of the above construction we consider the Lie group L_1 defined by the structure equations (3.4), which can be described in local coordinates $\{t, x, y, z, x_5, x_6, x_7\}$ as follows

$$\begin{aligned}
(4.27) \quad & e^1 = -dt, \\
& e^2 = \frac{1}{2} x_6 dx + \frac{1}{2} x_5 \cos x dy + \left(\frac{1}{2} x_6 \cos y + \frac{1}{2} x_5 \sin y \sin x\right) dz - \frac{1}{2} x_7 dt + \frac{1}{2} dx_7, \\
& e^3 = -\frac{1}{2} x_7 dx + \frac{1}{2} x_5 \sin x dy + \left(-\frac{1}{2} x_7 \cos y - \frac{1}{2} x_5 \sin y \cos x\right) dz - \frac{1}{2} x_6 dt + \frac{1}{2} dx_6, \\
& e^4 = \left(-\frac{1}{2} x_7 \cos x - \frac{1}{2} x_6 \sin x\right) dy - \frac{1}{2} \sin y (-x_6 \cos x + x_7 \sin x) dz - \frac{1}{2} x_5 dt + \frac{1}{2} dx_5, \\
& \eta_1 = e^5 = -x_6 dx + (-x_5 \cos x - 2 \sin x) dy \\
& \quad + (-x_6 \cos y - \sin y \sin x x_5 + 2 \sin y \cos x) dz + x_7 dt - dx_7, \\
& \eta_2 = e^6 = x_7 dx + (2 \cos x - x_5 \sin x) dy \\
& \quad + (x_7 \cos y + 2 \sin y \sin x + x_5 \sin y \cos x) dz + x_6 dt - dx_6, \\
& \eta_3 = e^7 = -2 dx + (\cos x x_7 + x_6 \sin x) dy \\
& \quad + (-2 \cos y + x_7 \sin y \sin x - x_6 \sin y \cos x) dz + x_5 dt - dx_5.
\end{aligned}$$

In this case $S = -\frac{1}{2}$ in (4.17). According to (4.26), the corresponding quaternionic Kähler metric is

$$(4.28) \quad g = \frac{1}{2b^2} (1 + \cosh \sigma) g_H + \frac{1}{16b^2} \sinh^2 \sigma (\eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{1}{b^2} d\sigma^2,$$

The quaternionic Kähler two forms are

$$F_i(\sigma) = \frac{1}{2b^2} (1 + \cosh \sigma) \omega_i + \frac{1}{16b^2} \sinh^2 \sigma \eta_j \wedge \eta_k - \frac{1}{2} \eta_i \wedge d\sigma.$$

The Ricci tensor is given by

$$Ric = -4b^2 g.$$

The metric (4.28) seems to be a new explicit quaternionic Kähler metric.

4.4.2. *Explicit quaternionic Kähler metrics from the zero-torsion qc-non-flat qc structure on \mathbf{I}_2 .* Consider the simply connected connected Lie group corresponding to the algebra \mathbf{I}_2 defined by the structure equations (3.10). This group can be described in local coordinates $(x, y, z, t, \varphi, \theta, \psi)$ as follows.

$$\begin{aligned}
(4.29) \quad & e_1 = -2 dt, \quad e_2 = dx - y dz + (-2x + zy) dt, \quad e_3 = dz - z dt, \quad e_4 = dy - y dt \\
& e_5 = -2 dx + 2 y dz - 2 (-2x + zy) dt - 4 \sin \psi d\theta + 4 \cos \psi \sin \theta d\varphi \\
& e_6 = -4 dz + 4 z dt - 4 \cos \psi d\theta - 4 \sin \psi \sin \theta d\varphi \\
& e_7 = -4 dy + 4 y dt - 4 d\psi - 4 \cos \theta d\varphi,
\end{aligned}$$

where θ, φ, ψ are the Euler angles, $0 < \theta < \pi$, $0 < \varphi < 2\pi$ and $0 < \psi < 4\pi$. Recall, in this case $S = -\frac{1}{4}$ in (4.17). According to (4.26), the corresponding quaternionic Kähler metric on $L_2 \times \mathbb{R}$ is

$$(4.30) \quad g = \frac{1}{2b^2} (1 + \cosh \sigma) g_H + \frac{1}{32b^2} \sinh^2 \sigma (\eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{2}{b^2} d\sigma^2.$$

The quaternionic Kähler 2-forms are expressed by

$$F_i(\sigma) = \frac{1}{2b^2} (1 + \cosh \sigma) \omega_i + \frac{1}{32b^2} \sinh^2 \sigma \eta_j \wedge \eta_k - \frac{1}{2} \eta_i \wedge d\sigma.$$

The Ricci tensor is given by

$$Ric = -2b^2 g.$$

The metric (4.30) seems to be a new explicit quaternionic Kähler metric.

4.5. Quaternionic Kähler metrics arising from a 3-Sasakian structure. Note that here the normalized scalar curvature is $S = 2$. The metric

$$g = ug_H + (u + au^2)((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{1}{4(u + au^2)}du^2$$

is a quaternionic Kähler, and in the case of $a = 0$ is the hyper Kähler cone over the 3-Sasakian manifold. These metrics have been found earlier in [37, Theorem 5.2].

4.6. Non quaternionic Kähler structures with closed four form in dimension 8. It is well known [33] in dimension $4n$, $n > 2$, the condition that the fundamental 4-form is closed is equivalent to the fundamental 4-form being parallel which is not true in dimension eight. Salamon constructed in [32] a compact example of an almost quaternion hermitian manifold with closed fundamental four form which is not Einstein, and therefore it is not quaternionic Kähler. We give below explicit non-compact example of that kind.

We consider seven dimensional qc structures with zero torsion endomorphism of the Biquard connection and constant qc scalar curvature S satisfying the structure equations

$$(4.31) \quad d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k.$$

Examples of such manifolds are provided by the following qc manifolds: i) the quaternionic Heisenberg group, where $S = 0$; ii) any 3-Sasakian manifold, where $S = 2$ (see [24] where it is proved that these structure equations characterize the 3-Sasakian qc manifolds); and iii) the zero torsion qc-flat group L_1 defined in Theorem 3.2 with the structure equations described in (3.4), where $S = -1/2$. Our example are inspired by the following Remark.

Remark 4.11. *In dimension seven, due to the relations $\omega_i \wedge \omega_j = 0$, $i \neq j$, a more general evolution than the one used in the proof of Theorem 4.9 can be considered, namely, let*

$$(4.32) \quad \omega_s(t) = f(t)\omega_s, \quad \eta_s(t) = f_s(t)\eta_s,$$

where f, f_1, f_2, f_3 are smooth function of t . Using the structure equations (4.31) one easily obtains that the equation $d\Omega = 0$ is satisfied and (4.6) is equivalent to the system

$$(4.33) \quad \begin{aligned} 3f' - 2(f_1 + f_2 + f_3) &= 0, \\ (ff_2f_3)' - Sf(f_1 - f_2 - f_3) - 6f_1f_2f_3 &= 0, \\ (ff_1f_3)' - Sf(-f_1 + f_2 - f_3) - 6f_1f_2f_3 &= 0, \\ (ff_1f_2)' - Sf(-f_1 - f_2 + f_3) - 6f_1f_2f_3 &= 0. \end{aligned}$$

On the other hand, $\langle F_1, F_2, F_3 \rangle$ is a differential ideal if and only if the following system holds

$$(4.34) \quad f(ff_jf_j)' - f'f_if_j + 2f_1f_2f_3 - 2f_if_j(f_i + f_j) + Sff_if_j - Sff_k = 0.$$

This claim is a consequence of the fact that working mod $\langle F_1, F_2, F_3 \rangle$ we have

$$dF_i = \frac{1}{f} (f(ff_jf_j)' - f'f_if_j + 2f_1f_2f_3 - 2f_i^2f_j - 2f_if_j^2 + Sff_if_j - Sff_k) \eta_j \wedge \eta_k \wedge dt.$$

Taking $f_1 = f_2 = f_3 = h$ in (4.33) we come to the case considered in Theorem 4.9.

The system (4.33) can be integrated completely when $S = 0$. This is achieved by introducing the new variable $du = f_1f_2f_3dt$, which allows to determine $ff_if_j = 6(u + a_k)$, where a_k is a constant. Thus

$$f_s = \frac{f}{6(u + a_s)} \frac{du}{dt}.$$

With the help of these three equations and the first equation of (4.33) we come to

$$\frac{9}{f} \frac{df}{dt} = \frac{du}{dt} \left(\frac{1}{u + a_1} + \frac{1}{u + a_2} + \frac{1}{u + a_3} \right),$$

hence

$$f^9 = C^9(u + a_1)(u + a_2)(u + a_3)$$

for some constant C . Now, the equations $ff_i f_j = 6(u + a_k)$ and the definition of u yield

$$f_i = \sqrt{\frac{6}{C}} \left(\frac{(u + a_j)^4 (u + a_k)^4}{(u + a_i)^5} \right)^{1/9}, \quad dt = (C/6)^{3/2} \frac{du}{((u + a_1)(u + a_2)(u + a_3))^{1/3}}.$$

when we impose also the system (4.34), in which we substitute $2(f_i + f_j) = 3f' - 2f_k$ and $f(ff_j f_j)' = 6f_1 f_2 f_3 - \frac{2}{3}(f_1 + f_2 + f_3) f_i f_j$ (using the equations of (4.33) and $\tau = 0$), we see that

$$3fdF_i = 10f_j f_k (2f_k - f_i - f_j) \eta_j \wedge \eta_k \wedge dt \quad \text{mod } \langle F_i, F_j, F_k \rangle.$$

Thus, $d\Phi = 0$ and $\langle F_1, F_2, F_3 \rangle$ is a differential ideal if and only if $f_1 = f_2 = f_3$ which yield the next Theorem.

Theorem 4.12. *The metric on the product of the seven dimensional quaternionic Heisenberg group with the real line defined by*

$$(4.35) \quad g = C ((u + a_1)(u + a_2)(u + a_3))^{1/9} (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + \\ \frac{6}{C} \left(\frac{(u + a_2)^8 (u + a_3)^8}{(u + a_1)^{10}} \right)^{1/9} (dx_5 + 2x_1 dx_2 + x_3 dx_4)^2 + \frac{6}{C} \left(\frac{(u + a_3)^8 (u + a_1)^8}{(u + a_2)^{10}} \right)^{1/9} (dx_6 + 2x_1 dx_3 + x_4 dx_2)^2 \\ + \frac{6}{C} \left(\frac{(u + a_1)^8 (u + a_2)^8}{(u + a_3)^{10}} \right)^{1/9} (dx_7 + 2x_1 dx_4 + x_2 dx_3)^2 + \left(\frac{C}{6} \right)^3 \frac{du^2}{((u + a_1)(u + a_2)(u + a_3))^{2/3}},$$

where a_1, a_2 and a_3 are three constants, not all of them equal to each other, supports an almost quaternion hermitian structure which has closed fundamental form, but is not quaternionic Kähler.

Remark 4.13. *Using MATHEMATICA one can check that the metrics (4.35) are Einstein exactly when $f_1 = f_2 = f_3$, in which case they are quaternionic Kähler metrics.*

We note that one of the arbitrary constants in (4.35) is unnecessary since a translation of the unknown u does not change the metric.

Let us also remark that the quaternionic Kähler metric (4.22) on the quaternionic Heisenberg group is obtained from the general family (4.35) by taking $\frac{6}{C^3} = b^2$ and $v = e^{2bt} = Cu^{1/3}$ when the constants are the same $a_1 = a_2 = a_3 = 2b$ and we use $u + a_1$ as a variable, which is denoted also by u .

4.7. Eight dimensional non quaternionic Kähler structures with fundamental forms generating a differential ideal. If one takes a solution of the system (4.34) which does not satisfy the system (4.33), one could obtain an almost quaternion hermitian structure such that $\langle F_1, F_2, F_3 \rangle$ is a differential ideal, yet, the structure is not of a quaternionic Kähler manifold (see also the paragraph after [27, Corollary 2.4]). For example, let $f \equiv 1$ and $S = 0$ in the system (4.34) and introduce the functions $u_i = \ln(f_j f_k)$. A small calculation turns the system (4.34) into the following equivalent system with six equations

$$(4.36) \quad f_i = e^{\frac{1}{2}(u_j + u_k - u_i)}, \quad f_i = \frac{1}{4} \left(\frac{du_j}{dt} + \frac{du_k}{dt} \right)$$

where as before (and in what follows) i, j, k denotes a positive permutation of 1, 2, 3. Thus we have to solve

$$\frac{d}{dt} (u_j + u_k) = 4e^{\frac{1}{2}(u_j + u_k - u_i)}$$

Now, we let $v_i = e^{-\frac{1}{2}u_i}$ so the above system becomes $\frac{d}{dt} (v_j v_k) = -2v_i$. Introducing $w_i = v_j v_k$ the latter equations take the form

$$\frac{d}{dt} w_i^2 = -4(w_1 w_2 w_3)^{1/2}.$$

Thus, the variables w_i^2 defer by additive constants, so we let $x = a_i - w_i^2$, $a_i = \text{const}$, where the variable x satisfies $dt = g(x)dx$ with

$$(4.37) \quad g(x) = \frac{1}{4} ((a_1 - x)(a_2 - x)(a_3 - x))^{-1/4}.$$

Solving back in terms of the wanted functions f_i we see that

$$(4.38) \quad f_i(x) = \frac{(a_j - x)^{1/4}(a_k - x)^{1/4}}{(a_i - x)^{3/4}}.$$

In conclusion, the evolution $\omega_i(x) = \omega_i$, $\eta_i(x) = f_i(x)\eta_i$ leads to the metric

$$g = g_H + f_1^2 \eta_1^2 + f_2^2 \eta_2^2 + f_3^2 \eta_3^2 + g(x)^2 dx^2,$$

where $F_i = \omega_i + f_j f_k \eta_j \wedge \eta_k - g f_i \eta_i \wedge dx$ and the functions f_i and g are defined in (4.37) and (4.38) respectively. The above metric supports an almost quaternion hermitian structure such that $\langle F_1, F_2, F_3 \rangle$ is a differential ideal but g is not a quaternionic Kähler metric. For the proof of the latter notice that if we set $f = 1$ and $S = 0$ the system (4.33) has no solution with nowhere vanishing functions f_i taking into account the first equation of the system.

5. $Sp(1)Sp(1)$ STRUCTURES AND $Spin(7)$ -HOLONOMY METRICS

An $Sp(1)Sp(1)$ structure on a seven dimensional manifold M^7 induces a G_2 -form ϕ by

$$(5.1) \quad \phi = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_3 - \eta_1 \wedge \eta_2 \wedge \eta_3.$$

Notice that the G_2 -form (5.1) is $Sp(1)Sp(1)$ invariant hence it is a well defined global form on the qc-manifold M . The Hodge dual $*\phi$ is

$$(5.2) \quad *\phi = \frac{1}{2}\omega_1^2 - (\omega_1 \wedge \eta_2 \wedge \eta_3 + \omega_2 \wedge \eta_3 \wedge \eta_1 + \omega_3 \wedge \eta_1 \wedge \eta_2).$$

Consider the family $(\eta_s(u), \omega_s(u))$ of $Sp(1)Sp(1)$ structures on M^7 depending on real parameter u , the corresponding G_2 form $\phi(u)$ and the $Spin(7)$ -form $\Psi(u)$ on $M^7 \times \mathbb{R}$ defined by [9]

$$(5.3) \quad \Psi(u) = F_1^- \wedge F_1^- + F_2^- \wedge F_2^- - F_3^- \wedge F_3^- = 2 \left[*\phi(u) - \phi(u) \wedge du \right],$$

where the 2-forms F_1^-, F_2^-, F_3^- are given by

$$(5.4) \quad F_1^- = \omega_1 - \eta_2 \wedge \eta_3 - \eta_1 \wedge du, \quad F_2^- = \omega_2 - \eta_3 \wedge \eta_1 - \eta_2 \wedge du, \quad F_3^- = \omega_3 + \eta_1 \wedge \eta_2 + \eta_3 \wedge du.$$

Following Hitchin, [19], the $Spin(7)$ -form $\Psi(u)$ is closed (and so [13] parallel with respect to the Levi-Civita connection) if and only if the G_2 structure is cocalibrated, $d*\phi = 0$, and the Hitchin flow equations $\partial_u(*\phi) = -d\phi$ are satisfied together with initial conditions $d*(\phi(u_0)) = 0$ at some point u_0 , i.e.

$$(5.5) \quad \partial_u(*\phi) = -d\phi, \quad d*(\phi(u_0)) = 0.$$

5.1. Construction of $Spin(7)$ -holonomy metrics using qc structures. At this point we shall assume that (M, g, \mathbb{Q}) is a qc manifold of dimension seven and investigate Hitchin's equations (5.5) leading to $Spin(7)$ -holonomy metrics.

Recall that in dimension seven the fundamental four form of the qc structure is given by $\Omega = 3\omega_1 \wedge \omega_1$. The latter together with Lemma 4.6, (5.2) and Theorem 4.7 yield the next Theorem.

Theorem 5.1. *Let (M, g, \mathbb{Q}) be seven dimensional qc manifold. The following conditions are equivalent.*

- a). *The fundamental four form is closed, $d\Omega = 0$;*
- b). *The G_2 -structure (5.1) is cocalibrated.*
- c). *The vertical distribution is integrable;*

Corollary 5.2. *The G_2 -structure (5.1) induced from a qc Einstein structure with constant qc scalar curvature is co-calibrated.*

Proof. The assumptions of the corollary lead to the structure equations (4.15) which imply $d\Omega = 0$ since in dimension seven $\omega_s \wedge \omega_t = \delta_{st} \text{vol.}|_H$. Now, Theorem 5.1 completes the proof. \square

Earlier [16] and [15] observed that every 3-Sasakian manifold has G_2 -forms, which are nearly parallel, and each one of them has a “squashing” which produces another nearly parallel G_2 -form. These structures are then used to obtain $Spin(7)$ metrics on the metric cone [3]. In [1] it is proven that every 3-Sasakian manifold has a “canonical” G_2 -form which is co-calibrated. From Corollary 5.2, a seven dimensional qc Einstein structure with constant qc scalar curvature has a co-calibrated G_2 -form which by [19] is a good candidate to construct a metric with holonomy contained in $Spin(7)$. It should be pointed out that a seven dimensional qc-Einstein structure with positive qc constant scalar curvature, locally, has a 3-Sasakian structure, see [20]. Nevertheless, the squashed metrics mentioned above are examples of seven dimensional qc Einstein structure with positive constant qc scalar curvature, so the quaternionic contact point of view allows a unified treatment of the construction.

We turn to the main result allowing the construction of $Spin(7)$ -holonomy metrics on the product of a qc manifold with a real interval.

Theorem 5.3. *Let (M, g, \mathbb{Q}) be a smooth qc Einstein manifold of dimension seven with constant normalized qc scalar curvature S . For a suitable constant a , the manifold $M \times \mathbb{R}$ has a metric with holonomy contained in $Spin(7)$ given by the following metric and $Spin(7)$ -form*

$$(5.6) \quad \begin{aligned} g &= u g_H + \frac{S u^{5/3} - 2a}{10 u^{2/3}} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{5 u^{2/3}}{18(S u^{5/3} - 2a)} du^2, \\ \Psi &= F_1^- \wedge F_1^- + F_2^- \wedge F_2^- - F_3^- \wedge F_3^-, \end{aligned}$$

where

$$(5.7) \quad \begin{aligned} F_1^-(u) &= u \omega_1 - \frac{S u^{5/3} - 2a}{10 u^{2/3}} \eta_2 \wedge \eta_3 - \frac{1}{6} \eta_1 \wedge du, \\ F_2^-(u) &= u \omega_2 - \frac{S u^{5/3} - 2a}{10 u^{2/3}} \eta_3 \wedge \eta_1 - \frac{1}{6} \eta_2 \wedge du, \\ F_3^-(u) &= u \omega_3 + \frac{S u^{5/3} - 2a}{10 u^{2/3}} \eta_1 \wedge \eta_2 + \frac{1}{6} \eta_3 \wedge du. \end{aligned}$$

Proof. Let h and f be some functions of the unknown t . Consider the 2-forms defined by

$$(5.8) \quad \begin{aligned} F_1^-(t) &= f(t) \omega_1 - h^2(t) \eta_2 \wedge \eta_3 - h(t) \eta_1 \wedge dt, \\ F_2^-(t) &= f(t) \omega_2 - h^2(t) \eta_3 \wedge \eta_1 - h(t) \eta_2 \wedge dt, \\ F_3^-(t) &= f(t) \omega_i + h^2(t) \eta_1 \wedge \eta_2 + h(t) \eta_3 \wedge dt. \end{aligned}$$

Substituting (5.8) in (5.3), then taking the exterior derivative of the obtained 4-form while applying (4.7) and (4.15) yields

$$(5.9) \quad \begin{aligned} d\Psi(t) &= (2f f' - 12fh) \omega_1 \wedge \omega_1 \wedge dt \\ &\quad - \left(2(fh^2)' - 2fhS - 4h^3 \right) (\omega_1 \wedge \eta_2 \wedge \eta_3 + \omega_2 \wedge \eta_3 \wedge \eta_1 + \omega_3 \wedge \eta_1 \wedge \eta_2) \wedge dt. \end{aligned}$$

Hence, (5.9) shows that the condition $d\Psi(t) = 0$ is equivalent to the ODE system

$$(5.10) \quad 3f f'' + (f')^2 - 9Sf = 0, \quad h = \frac{1}{6} f'.$$

To solve this differential equation, we use $v = f^{4/3}$ as a variable. Equation (5.10) shows that

$$\left(\frac{dv}{dt} \right)^2 = \frac{32(Sv^{5/4} - 2a)}{5},$$

where a is a constant. Hence, $\left(\frac{dt}{df} \right)^2 = \left(\frac{dt}{dv} \right)^2 \left(\frac{dv}{df} \right)^2 = \frac{5f^{2/3}}{18(Sf^{5/3} - 2a)}$, which implies

$$h^2 = \frac{1}{36} (f')^2 = \frac{Sf^{5/3} - 2a}{10f^{2/3}}.$$

Renaming f to u gives the metric and the $Spin(7)$ form ψ in the form (5.6) together with (5.7). \square

5.2. $\text{Spin}(7)$ -holonomy metrics based on qc Einstein structure with zero qc scalar curvature.

5.2.1. *Spin(7) holonomy metrics from the quaternionic Heisenberg group.* Consider the 7-dimensional quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ with structure equations (3.1) taken for $n = 1$ equipped with its standard qc structure. The corresponding eight dimensional $\text{Spin}(7)$ -holonomy metric written in Theorem 5.3 can be written in the form

$$(5.11) \quad g = u^3 ((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{a^2}{16} u^{-2} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{4}{a^2} u^6 du^2.$$

These $\text{Spin}(7)$ -holonomy metrics are found in [17, Section 4.3.1].

5.2.2. *New Spin(7)-holonomy metrics from the quaternionic Heisenberg group.* New $\text{Spin}(7)$ -holonomy metrics can be obtained similarly to the derivation of (4.35). We evolve the structure as in (4.32), namely $\omega_s(u) = f(u)\omega_s$, $\eta_s(u) = f_s(u)\eta_s$. Using the structure equations of the quaternionic Heisenberg group, $d\eta_s = 2\omega_s$, one easily obtains that the second equation of the (5.5) is equivalent to the system

$$(5.12) \quad \begin{aligned} f' - 2(f_1 + f_2 + f_3) &= 0, & (ff_2f_3)' - 2f_1f_2f_3 &= 0, \\ (ff_1f_3)' - 2f_1f_2f_3 &= 0, & (ff_1f_2)' - 2f_1f_2f_3 &= 0. \end{aligned}$$

We integrate the system (5.12) to obtain the next family of $\text{Spin}(7)$ -holonomy metrics which seems to be new

$$(5.13) \quad g = C ((u + a_1)(u + a_2)(a_3 - u)) ((dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + \frac{2}{C} \frac{1}{(u + a_1)^2} (dx_5 + 2x_1dx_2 + 2x_3dx_4)^2 + \frac{2}{C} \frac{1}{(u + a_2)^2} (dx_6 + 2x_1dx_3 + 2x_4dx_2)^2 + \frac{2}{C} \frac{1}{(a_3 - u)^2} (dx_7 + 2x_1dx_4 + 2x_2dx_3)^2 + \frac{C^3}{8} (u + a_1)^2 (u + a_2)^2 (a_3 - u)^2 du^2).$$

Taking $a_2 = -a_3 = a_1$ into (5.13) one gets the $\text{Spin}(7)$ -holonomy metrics (5.11). Since the coefficients of the metrics (5.13) are continuous with respect to the parameters, and since the holonomy is equal to $\text{Spin}(7)$ for $(a_2, a_3) = (a_1, -a_1)$ then the same holds for any (a_2, a_3) in an small neighbourhood of $(a_1, -a_1)$. Thus, we get a three parameter family of metrics with holonomy equal to $\text{Spin}(7)$ which seem to be new.

More generally, for any triple a_1, a_2, a_3 of real numbers one can find an open interval $J \subset \mathbb{R}$ such that for $u \in J$ the holonomy of the metrics (5.13) equals $\text{Spin}(7)$.

5.3. $\text{Spin}(7)$ -holonomy metrics based on qc Einstein structure with negative scalar curvature.

5.3.1. *Explicit Spin(7)-holonomy metrics from the zero torsion qc-flat structure on \mathfrak{l}_1 .* We consider the Lie group L_1 defined by the structure equations (3.4) which can be described in local coordinates with (4.27). In this case $S = -\frac{1}{2}$ according to Theorem 3.2. The corresponding metric with holonomy contained in $\text{Spin}(7)$ from Theorem 5.3 has the form (taking $b = -4a$)

$$(5.14) \quad g = u ((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{(b - u^{5/3})}{20u^{2/3}} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{5u^{2/3}}{9(b - u^{5/3})} du^2.$$

and seem to be new.

Calculating the curvature of the metrics (5.14), using the local coordinates (4.27) of the group, one finds that there are at least 16 independent curvature forms which implies that the holonomy of these metrics is equal to $\text{Spin}(7)$.

5.3.2. *Explicit Spin(7)-holonomy metrics from the zero torsion qc-non-flat structure on \mathfrak{l}_2 .* We consider the Lie group L_2 defined by the structure equations (3.10) which can be described in local coordinates with (4.29). In this case $S = -\frac{1}{4}$ according to Theorem 3.3. The corresponding metric with holonomy contained in $\text{Spin}(7)$ from Theorem 5.3 takes the form given by equation (5.6) with $S = -\frac{1}{4}$ and $b = -8a$,

$$(5.15) \quad g = u ((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{(b - u^{5/3})}{40u^{2/3}} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{10u^{2/3}}{9(b - u^{5/3})} du^2.$$

and seems to be a new metric with holonomy equal to $Spin(7)$. The latter fact can be seen by a direct calculation applying the local coordinates (4.29) to (5.15) and showing that the curvature 2-forms span a space of dimension twenty one.

5.4. $Spin(7)$ -holonomy metrics from a 3-Sasakian manifold. This case was investigated in general in [4] and explicit solutions in particular cases are known (see [4] and references therein). We use again only the particular solution to (5.12) found above. The metric with holonomy contained in $Spin(7)$ from Theorem 5.3 takes the form given by equation (5.6) with $S = 2$,

$$g = u \left((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 \right) + \frac{u^{5/3} - a}{5u^{2/3}} \left((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2 \right) + \frac{5u^{2/3}}{36(u^{5/3} - a)} du^2.$$

This family includes the (first) complete metric with holonomy $Spin(7)$ constructed by Bryant and Salamon on the total space of the spin bundle over the sphere S^4 [10, 18], see also [4, p. 6].

REFERENCES

- [1] Agricola, I., & Friedrich, Th., *3-Sasakian manifolds in dimension seven, their spinors and G_2 -structures*. J. Geom. Phys. 60 (2010), no. 2, 326–332. 20
- [2] Alekseevsky, D. & Kamishima, Y., *Pseudo-conformal quaternionic CR structure on $(4n + 3)$ -dimensional manifold*, Ann. Mat. Pura Appl. **187** (2008), 487–529; math.GT/0502531. 4, 11
- [3] Bär, C., *Real Killing spinors and holonomy*. Comm. Math. Phys. 154 (1993), no. 3, 509–521. 20
- [4] Bazaikin, Y., *On the new examples of complete noncompact $Spin(7)$ -holonomy metrics*, Siberian Math. J., **48** (2007), 8–25. 22
- [5] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). 2, 3, 4, 5
- [6] Biquard, O., *Quaternionic contact structures*, Quaternionic structures in mathematics and physics (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999. 2
- [7] Boyer, Ch. & Galicki, K., *3-Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, 123–184, Surv. Differ. Geom., **VI**, Int. Press, Boston, MA, 1999. 5
- [8] Boyer, Ch., Galicki, K. & Mann, B., *The geometry and topology of 3-Sasakian manifolds*, J. Reine Angew. Math., **455** (1994), 183–220. 5
- [9] Bryant, R., Harvey, R., *Submanifolds in hyper-Kähler manifolds*, J. Am. Math. Soc. **2** (1989), 1–31. 19
- [10] Bryant, R., Salamon, S., *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850. 22
- [11] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 851–885. 4
- [12] Duchemin, D., *Quaternionic contact hypersurfaces*, math.DG/0604147. 2
- [13] Fernández, M., *A classification of Riemannian manifolds with structure group $Spin(7)$* , Ann. Mat. Pura Appl. (4) **143** (1986), 101–122. 19
- [14] Folland, G.B., Stein, E.M., *Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429–522. 2
- [15] Friedrich, T., Kath, I., Moroianu, A., & Semmelmann, U., *On nearly parallel G_2 -structures*, J. Geom. Phys. 23 (1997), no. 3–4, 259–286. 20
- [16] Galicki, K., & Salamon, S., *Betti numbers of 3-Sasakian manifolds*, Geom. Dedicata 63 (1996), no. 1, 45–68. 20
- [17] Gibbons, G. W.; Lü, H.; Pope, C. N.; Stelle, K. S., *Supersymmetric domain walls from metrics of special holonomy*. Nuclear Phys. B 623 (2002), no. 1–2, 3–46. 3, 15, 21
- [18] Gibbons, G.W., Page, D.N., Pope, C.N., *Einstein metrics on S^3 , R^3 and R^4 bundles*, Commun. Math. Phys. **127** (1990), 529–553. 22
- [19] Hitchin, N., *Stable forms and special metrics*. In Global Differential Geometry: In Global Differential Geometry: The Mathematical Legacy of Alfred Gray (Bilbao, 2000) volume 288 of Contemp. Math., American Math. Soc., Providence, RI, 2001, 70–89. 19, 20
- [20] Ivanov, S., Minchev, I., & Vassilev, D., *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, preprint, math.DG/0611658. 2, 3, 4, 5, 6, 7, 13, 15, 20
- [21] Ivanov, S., Minchev, I., & Vassilev, *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem*, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 4, 1041–1067. 2, 5
- [22] Ivanov, S., Minchev, I., & Vassilev, in preparation. 2
- [23] Ivanov, S., & Vassilev, D., *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pures Appl. **93** (2010), 277–307. 2, 3, 5, 6, 13
- [24] Ivanov, S., & Vassilev, D., *Quaternionic contact manifolds with a closed fundamental 4-form*, to appear in Bull. London Math. Soc., arXiv:0810.3888. 2, 7, 12, 13, 17

- [25] Kashiwada, T., *A note on Riemannian space with Sasakian 3-structure*, Nat. Sci. Reps. Ochanomizu Univ., **22** (1971), 1–2. 5
- [26] Kuo, Y.-Y., *On almost contact 3-structures*, Tohoku Math. J. **22**(1970), 325–332. 10
- [27] Maciá, O., *A Nearly Quaternionic Structure on $SU(3)$* , arXiv:0908.4183 3, 18
- [28] Mostow, G. D., *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp. 2
- [29] Nesterenko, M. O., & Boyko, V. M., *Realizations of indecomposable solvable 4-dimensional real Lie algebras*. Symmetry in nonlinear mathematical physics, Part 1, 2 (Kyiv, 2001), 474–477, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 43, Part 1, 2, Natsi-onal. Akad. Nauk Ukrani, I-nst. Mat., Kiev, 2002. 8, 9
- [30] O’Neill, B., *Semi-Riemannian geometry. With applications to relativity*. Pure and Applied Mathematics, 103. Academic Press, Inc., New York, 1983. 14
- [31] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) 129 (1989), no. 1, 1–60. 2
- [32] Salamon, S., *Almost parallel structures*. In Global Differential Geometry: The Mathematical Legacy of Alfred Gray (Bilbao, 2000) volume 288 of Contemp. Math., American Math. Soc., Providence, RI, 2001, 162–181. 3, 11, 17
- [33] Swann, A., *Quaternionic Kähler geometry and the fundamental 4-form*, Proceedings of the Workshop on Curvature Geometry (Lancaster, 1989), 165–173, ULDM Publ., Lancaster, 1989. 3, 11, 14, 17
- [34] Swann, A., *HyperKähler and quaternionic Kähler geometry*, Math. Ann. vol 289 (1991), 421–450. 11
- [35] Strichartz, R. S., *Sub-Riemannian geometry*. J. Differential Geom. 24 (1986), no. 2, 221–263. 15
- [36] Wang, W., *The Yamabe problem on quaternionic contact manifolds*, Ann. Mat. Pura Appl., **186** (2007), no. 2, 359–380. 2
- [37] Watanabe, Y., & Mori, H., *From Sasakian 3-structures to quaternionic geometry*. Arch. Math. (Brno) 34 (1998), no. 3, 379–386. 12, 17

(Luis C. de Andrés, Marisa Fernández, José A. Santisteban) UNIVERSIDAD DEL PAÍS VASCO, FACULTAD DE CIENCIA Y TECNOLOGÍA, DEPARTAMENTO DE MATEMÁTICAS, APARTADO 644, 48080 BILBAO, SPAIN

E-mail address: `luisc.deandres@ehu.es`

E-mail address: `marisa.fernandez@ehu.es`

E-mail address: `joseba.santisteban@ehu.es`

(Stefan Ivanov) UNIVERSITY OF SOFIA, FACULTY OF MATHEMATICS AND INFORMATICS, BLVD. JAMES BOURCHIER 5, 1164, SOFIA, BULGARIA

AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104-6395

E-mail address: `ivanovsp@fmi.uni-sofia.bg`

(Luis Ugarte) DEPARTAMENTO DE MATEMÁTICAS-I.U.M.A., UNIVERSIDAD DE ZARAGOZA, CAMPUS PLAZA SAN FRANCISCO, 50009 ZARAGOZA, SPAIN

E-mail address: `ugarte@unizar.es`

(Dimitar Vassilev) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO, 87131-0001

E-mail address: `vassilev@math.unm.edu`